Surfaces in the complex projective plane
and their mapping class groups

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Abstract An orientation preserving diffeomorphism over a surface embedded in a 4-manifold is called extendable, if this diffeomorphism is a restriction of an orientation preserving diffeomorphism on this 4-manifold. In this paper, we investigate conditions for extendability of diffeomorphisms over surfaces in the complex projective plane.

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Dedicated to Professor Yukio Matsumoto for his 60th birthday

1 Introduction

There are deformations of embedded surfaces in 4-manifolds which induce isotopically non-trivial diffeomorphisms on surfaces. We introduce two typical examples.

For the first example, we consider a deformation of an annulus embedded in $S^3 \times [-1,1]$ so that, under this deformation, the boundary of this annulus is fixed. Let $S^1 \times [0,1]$ be an annulus embedded in $S^3 \times \{0\} \subset S^3 \times [-1,1]$, and $t: S^3 \times [-1,1] \rightarrow [-1,1]$ a projection to the second factor. We deform $S^1 \times [0,1]$ as in Figure 1. First, we isotope $S^1 \times [0,1]$ in $S^3$ from (1) to (3). Next, we isotope $S^1 \times [0,1]$ so that outside of the annulus $A$ of (3) $t = 0$, and inside $t > 0$. Then we isotope $S^1 \times [0,1]$ inside $A$ so that, when we push $A$ down to $S^3 \times \{0\}$, $S^1 \times [0,1]$ is as in (4). Finally, we isotope $S^1 \times [0,1]$ in $S^3$ from (4) to (6). The composition of these deformations induce a square of Dehn twist about the core circle $S^1 \times \{\frac{1}{2}\}$ of $S^1 \times [0,1]$.

For the second example, we consider a deformation of a non-singular plane curve of degree 3. A torus is defined as a quotient of the complex plane by a lattice $\mathbb{Z} + \mathbb{Z} \sqrt{-1}$. We embed this torus into the complex projective plane.
by using the Weierstrass $\wp$ function associated to this lattice, then the image of this embedding is a non-singular plane curve of degree 3. We deform this lattice, $\mathbb{Z} + \mathbb{Z}(\sqrt{-1} + t)$, where $0 \leq t \leq 1$ is a parameter of this deformation. Then the embedding is deformed isotopically and, finally (when $t = 1$), brought back to the original position. This deformation induces a Dehn twist on the non-singular plane curve of degree 3.

In this paper, we investigate a topological meaning of the above phenomena.

We settle a general formulation. Let $M$ be a simply connected compact oriented smooth 4-manifold (possibly with boundary) and $F$ be a compact oriented smooth 2-manifold (possibly with boundary) embedded in $M$. We call the pair $(M,F)$ a knotted surface. In particular, if $F$ is characteristic, that is to say, $F \cdot X \equiv X \cdot X \mod 2$ for any $X \in H_2(M,\mathbb{Z})$, then we call this pair $(M,F)$ a knotted characteristic surface. An orientation preserving diffeomorphism $\psi$ over $F$ is extendable if there is an orientation preserving diffeomorphism $\Psi$ over $M$ such that $\Psi|_F = \psi$. In general, for an oriented manifold $A$ and its submanifold $B$, we denote

$$ \text{Diff}_+(A, \text{fix } B) = \left\{ \psi \middle| \text{an orientation preserving diffeomorphism over } A \right\} $$

such that $\psi|_B = id_B$.

If $B = \phi$, we denote this group by $\text{Diff}_+(M)$. The group $\pi_0(\text{Diff}_+(F, \text{fix } \partial F))$ is called the mapping class group of $F$ and denoted by $\mathcal{M}_F$. If $F$ is a closed oriented surface of genus $g$, this group is denoted by $\mathcal{M}_g$. We define

$$ \mathcal{E}(M,F) = \{ \psi \in \mathcal{M}_F \mid \psi \text{ is extendable} \}. $$

**Figure 1**

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This is a subgroup of $\mathcal{M}_F$ and is a central object of this paper.
In the case where $M = S^4$, there are several works on this group. Let $(S^4, \Sigma_g)$ be the genus $g$ trivial knotted surface in $S^4$. When $g = 1$, Montesinos \cite{19} investigated $\mathcal{E}(S^4, \Sigma_1)$, and when $g \geq 2$, the author \cite{11} investigated $\mathcal{E}(S^4, \Sigma_g)$. Let $(S^3, k)$ be a knot in $S^3$ and $(S^4, S(k))$ (resp. $(S^4, \tilde{S}(k))$) the spun (resp. the twisted spun) of $(S^3, k)$. When $(S^3, k)$ is a torus knot, Iwase \cite{13} investigated $\mathcal{E}(S^4, S(k))$ and $\mathcal{E}(S^4, \tilde{S}(k))$, and when $(S^3, k)$ is an arbitrary knot, the author \cite{10} investigated these groups.
In this paper, we investigate the case where $M$ is the complex projective plane $\mathbb{CP}^2$. In §3, we treat the case where $(\mathbb{CP}^2, \Sigma_g)$ is a standard embedding of $\Sigma_g$. In §4 we treat the case where $(\mathbb{CP}^2, F)$ is a non-singular plane curve. From §5 to the end of this paper, we treat the case where $(\mathbb{CP}^2, F)$ is a connected sum of a non-singular plane curve of degree 3 and a trivial embedding.

2 Preliminary: A Hopf band on the boundary of the 4-ball

A link $L$ in $S^3$ is called a fibered link if there is a map $\phi: S^3 \setminus L \to S^1$ which is a fiber bundle projection. For each $t \in S^1$, $\phi^{-1}(t) = F$, which does not depend on $t$, is called the fiber of $L$. Since $\phi$ is a bundle projection, $S^3 \setminus L$ is diffeomorphic to the quotient of $F \times [0, 1]$ by an equivalence $(x, 0) \sim (h(x), 1)$ where $h$ is a diffeomorphism over $F$ and called the monodromy of $L$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{hopf_band.png}
\caption{Positive Hopf band (left) and Negative Hopf band (right)}
\end{figure}

A Hopf band is an annulus embedded in $S^3$ as in Figure 2. In this picture, there are two types of Hopf bands. In this note, we treat both types of Hopf bands. The boundary of a Hopf band is called a Hopf link. The Hopf link is a fibered link whose fiber is the Hopf band and whose monodromy is a Dehn twist about the core circle of the Hopf band. Let $B$ be a Hopf band in $S^3$ which is a boundary of a 4-ball $D^4$. We push the interior of $B$ into the interior of $D^4$ and let $B'$ be the annulus obtained by this deformation and let $c$ be the core circle of $B'$.
Proposition 2.1  The Dehn twist $T_c$ about $c$ is extendable, i.e. there is an element $T \in \Diff_+(D^4, \fix \partial D^4)$ such that $T|_{\partial B} = T_c$.

Proof  Since $\partial B$ is a fibered link, whose fiber is $B$ and whose monodromy is $T_c$, there is an orientation preserving diffeomorphism $\psi$ of $\partial D$ which is defined by shifting fibers. Let $\psi$ be a diffeomorphism defined as follows

$$T|_{N(\partial D^4)}(x, t) = \begin{cases} (\psi_t(x), t) & 0 \leq t \leq 1 \\ (\psi_{2-t}(x), t) & 1 \leq t \leq 2 \end{cases}$$

$$T|_{D^4 \setminus N(\partial D^4)} = \text{id.}$$

This is the diffeomorphism which we need.

Remark 2.2  Let $(S^4, \Sigma_g)$ be the genus $g$ surface standardly embedded in $S^4$. In [11], the author showed that $T_{c_4}T_{c_3}T_{c_4}^{-1} \in \mathcal{E}(S^4, \Sigma_g)$ by using Montesinos’ result [19] Theorem 5.3 ($c_3$ and $c_4$ are as in Figure 7). We show this fact by using Proposition 2.1. The 4-sphere $S^4$ is constructed from two 4-balls $D^4_+$, $D^4_-$ with attaching along the boundary $S^3 = \partial D^4_+ = \partial D^4_-$. We parametrize the regular neighborhood $N(\partial D^4_+) = S^3 \times [0, 2]$ in $D^4_+$ so that $\partial D^4_+ = S^3 \times \{0\}$. The regular neighborhood $N$ of $T_{c_4}(c_3)$ in $\Sigma_g$ is a Hopf band in $S^3 \subset S^4$. We push the interior of $N$ into the interior of $D^4_+$, then we get an annulus $N'$ properly embedded in $D^4_+$. We may assume, by the above parametrization of $N(\partial D^4_+)$, $N' \cap S^3 \times \{t\} = \partial N \times \{t\}$ for $0 \leq t < 2$ and $N' \cap S^3 \times \{2\} = N \times \{2\}$. We denote $D^4_+ \setminus S^3 \times [0, 1]$ by $D'$. By applying Proposition 2.1 to $(D', N' \cap D')$, we show that there is an element $T \in \Diff_+(D', \fix \partial D')$ such that $T|_{N' \cap D'} = T_{c_4}T_{c_3}T_{c_4}^{-1}$. Therefore, we see $T_{c_4}T_{c_3}T_{c_4}^{-1} \in \mathcal{E}(S^4, \Sigma_g)$.

3  Surfaces standardly embedded in the complex projective plane

For the free action of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on $\mathbb{C}^3 \setminus \{(0, 0, 0)\}$ defined by $\lambda(z_0, z_1, z_2) = (\lambda z_0, \lambda z_1, \lambda z_2)$, we take the quotient $\mathbb{C}\mathbb{P}^2 = (\mathbb{C}^3 \setminus \{(0, 0, 0)\})/\mathbb{C}^*$. This space $\mathbb{C}\mathbb{P}^2$ is a closed oriented 4-manifold and called the complex projective plane. This 4-manifold $\mathbb{C}\mathbb{P}^2$ is constructed from $D^4$ by attaching a 2-handle $h^2$ along the frame 1 trivial knot $K_0$ in $\partial D^4$, and attaching a 4-handle $h^4$. A 3-dimensional handlebody $H_g$ is an oriented 3-manifold which is constructed from
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a 3-ball with attaching \(g\) 1-handles. Any image of embeddings of \(H_g\) into \(\mathbb{CP}^2\) are isotopic each other. Therefore, \((\mathbb{CP}^2, \partial H_g)\) is unique. A surface standardly embedded in \(\mathbb{CP}^2\) is \((\mathbb{CP}^2, \partial H_g)\). We obtain:

**Theorem 3.1** For any \(g\), \(\mathcal{E}(\mathbb{CP}^2, \partial H_g) = \mathcal{M}_g\).

**Proof** Let \(D^4\) be the 4-ball used to construct \(\mathbb{CP}^2\) and \(N(\partial D^4)\) be the regular neighborhood of \(\partial D^4\) in \(D^4\). We parametrize \(N(\partial D^4) = S^3 \times [0, -1]\), so that \(S^3 \times \{0\} = \partial D^4\) and, for \(-1 \leq t < 0\), \(S^3 \times \{t\}\) is in the interior of \(D^4\). Since the image of embedding of \(H_g\) in \(\mathbb{CP}^2\) is unique up to isotopy, we assume that \(H_g \subset S^3 \times \{-1\}\) and that each simple closed curve \(c\) on \(H_g\) which corresponds to Lickorish generator of mapping class group \(\mathcal{M}_g\) is a trivial knot in \(S^3 \times \{-1\}\). The regular neighborhood \(N(c)\) of \(c\) on \(\partial H_g\) is an annulus trivially embedded in \(S^3 \times \{-1\}\). At first, we deform \(H_g\) in \(S^3 \times \{-1\}\) so that, if we forget the second factor \([0, -1]\), \(c \cup K_0\) becomes a Hopf link in \(S^3\). We push \(N(c)\) into \(\partial(D^4 \cup h^2)\), then \(N(c)\) becomes a Hopf band in \(\partial h^4\). By applying Proposition 2.1, we see that \(T_c\) is extendable in \(h^4\), and so in \(\mathbb{CP}^2\). \(\square\)

### 4 Non-singular plane curves

We review here the topological description of non-singular plane curves by Akbulut and Kirby \[1\] (see also [6, 6.2.7]).

![Figure 3](image-url)
An \((m,n)\)-torus link \(T_{m,n}\) is an oriented link in \(S^3 = \partial D^4\) consisting of \(\gcd(m,n)\) oriented circles in the boundary of the tubular neighborhood \(U\) of the trivial knot, representing \(m\mu + n\lambda\) in \(H_1(\partial U; \mathbb{Z})\), where \(\mu = [\text{the meridian of the trivial knot}]\) and \(\lambda = [\text{the longitude of the trivial knot}]\). There is a canonical Seifert surface \(F_{m,n}\) for \(T_{m,n}\), consisting of \(n\) disks connected by \(m(n-1)\) twisted bands as in Figure 3. As \(K_0\), we take a trivial knot given by pushing \(T_{1,0}\) into the complement of \(U\) (see the left hand side of Figure 4). From here, we consider only the case where \(m = n = d\). As shown in the right hand side of Figure 4, \(T_{d,d}\) becomes \(d\) components trivial link in \(\partial (D^4 \cup h^2)\). Let \(D_d\) be disjoint 2-disks in \(\partial (D^4 \cup h^2)\) which bound this trivial link.

Let \(K_d\) be a non-singular plane curve of degree \(d\), then \(K_d\) is a genus \(\frac{(d-1)(d-2)}{2}\) closed oriented surface embedded in \(\mathbb{CP}^2\). We remark that \(K_d\) is unique up to isotopy, \(K_d = \{[X : Y : Z] \in \mathbb{CP}^2 | X^d + Y^d + Z^d = 0\}\) and \([K_d] = d[\mathbb{CP}^1] \in H_2(\mathbb{CP}^2; \mathbb{Z})\). Akbulut and Kirby showed:

**Proposition 4.1** \(K_d = F_{d,d} \cup D_d\).

Thus we obtain:

**Theorem 4.2** When \(d = 3,4\), \(\mathcal{E}(\mathbb{CP}^2, K_d) = \mathcal{M}_{g_d}\), where \(g_d = \frac{(d-1)(d-2)}{2}\).

**Proof** When \(d = 3\), \(K_3\) is homeomorphic to a 2-dimensional torus \(T^2\). In \(F_{3,3}\) (see Figure 3), each regular neighborhood of \(c_1\) and \(c_2\) is a Hopf band. Therefore, by Proposition 2.1, \(T_{c_1}\) and \(T_{c_2}\) are elements of \(\mathcal{E}(\mathbb{CP}^2, K_3)\). On the other hand, \(T_{c_1}\) and \(T_{c_2}\) generate \(\mathcal{M}_1\). Hence, \(\mathcal{E}(\mathbb{CP}^2, K_3) = \mathcal{M}_1\).
When \( d = 4 \), we do the same as the above case. We remark that the Dehn twists about the simple closed curves in Figure 5 corresponding to the simple closed curves in \( F_{4,4} \) (see Figure 3) with the same symbols generate the mapping class group of genus 3 surface \([16]\).

When \( d \geq 5 \), \( E(\mathbb{C}P^2, K_d) \) is unknown. It is, however, not the case that \( E(\mathbb{C}P^2, K_d) = \mathcal{M}_g \), because, when \( d \) is odd, \( K_d \) is a characteristic surface, so the Rokhlin quadratic form on \( H_1(K_d; \mathbb{Z}_2) \) is well-defined (we review the definition of the Rokhlin quadratic form in the next section). By the definition of the Rokhlin quadratic form, if a diffeomorphism on \( K_d \) is extendable to \( \mathbb{C}P^2 \), this diffeomorphism should preserve this form. Hence:

**Theorem 4.3** When \( d \) is an odd integer greater than or equal to 5, \( E(\mathbb{C}P^2, K_d) \) is a proper subgroup of \( \mathcal{M}_g \), where \( g_d = \frac{(d-1)(d-2)}{2} \).

5 Connected sum of the non-singular plane curve of degree 3 and trivial knotted surface

We define knotted surfaces investigated from here to the end of this paper. The images of any embeddings of a 3-dimensional handlebody \( H_g \) into \( S^4 \) are isotopic each other. We call this \( \Sigma_g \)-knot \((S^4, \partial H_g)\) a trivial \( \Sigma_g \)-knot, and this is denoted by \((S^4, \Sigma_g)\). Let \((\mathbb{C}P^2, K_3)\) be a nonsingular cubic plane curve. We define connected sum of \((\mathbb{C}P^2, K_3)\) and \((S^4, \Sigma_{g-1})\) following the construction by Boyle [3] as follows. We choose points \( p \) and \( q \) on \( K_3 \) and \( \Sigma_{g-1} \) respectively, and find small 4-balls \( B_1 \) and \( B_2 \) centered at \( p \) and \( q \) such that the pairs \((B_1, B_1 \cap K_3)\) and \((B_2, B_2 \cap \Sigma_{g-1})\) are equivalent to the standard pair \((B^4, B^3)\). Now we glue the pairs \((S^4 \setminus \text{int}(B_1), K_3 \setminus \text{int}(B_1))\) and \((\mathbb{C}P^2 \setminus \text{int}(B_2), \Sigma_{g-1} \setminus \text{int}(B_2))\) together by an orientation-reversing diffeomorphism \( f : \partial B_1 \to \partial B_2 \) such that \( f(\partial B_1 \cap K_3) = \partial B_2 \cap \Sigma_{g-1} \). Since the connected sum of \( \mathbb{C}P^2 \) and \( S^4 \) is diffeomorphic to \( \mathbb{C}P^2 \), we get a surface in \( \mathbb{C}P^2 \) and denote this characteristic
knotted surface by \((\mathbb{C}P^2, K_3\#\Sigma_{g-1})\). From here to the end of this paper, we investigate on the group \(\mathcal{E}(\mathbb{C}P^2, K_3\#\Sigma_{g-1})\).

For a knotted characteristic surface \((M,F)\), where \(M\) is a simply connected smooth closed oriented 4-manifold, we define a quadratic form \((the \, Rokhlin \, quadratic \, form) \, q_F : H_1(F;\mathbb{Z}_2) \to \mathbb{Z}_2\): Let \(P\) be a compact surface embedded in \(M\), with its boundary contained in \(F\), normal to \(F\) along its boundary, and its interior is transverse to \(F\). Let \(P'\) be a surface transverse to \(P\) obtained by sliding \(P\) parallel to itself over \(F\). Define \(q_F(\partial P) = \#(\text{int} \, P \cap (P' \cup F)) \mod 2\). This is a well-defined quadratic form with respect to the \(\mathbb{Z}_2\)-homology intersection form \((,)_2\) on \(F\), i.e. for each pair of elements \(x, y\) of \(H_1(F;\mathbb{Z}_2)\), \(q_F(x+y) = q_F(x) + q_F(y) + (x,y)_2\). By the definition of the Rokhlin quadratic from \(q_F\), if \(\psi \in \text{Diff}_+(F)\) is extendable, then \(\psi\) preserves \(q_F\), that is to say, \(q_F(\psi(x)) = q_F(x)\) for any \(x \in H_1(F;\mathbb{Z}_2)\). We will show,

**Theorem 5.1** For any \(g \geq 2\),

\[
\mathcal{E}(\mathbb{C}P^2, K_3\#\Sigma_{g-1}) = \left\{ \psi \in \mathcal{M}_g \mid \begin{array}{c}
q_{K_3\#\Sigma_{g-1}}(\psi_*(x)) = q_{K_3\#\Sigma_{g-1}}(x) \\
\text{for any } x \in H_1(K_3\#\Sigma_{g-1};\mathbb{Z}_2)
\end{array} \right\}.
\]

In §6 we investigate on a system of generators for the right hand side group in the equation of Theorem 5.1. In §7 we show that each element of this system of generators is extendable.

6 A finite set of generators for the odd spin mapping class group

Figure 6

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We settle some notations. Let \( P_g \) be a planar surface constructed from a 2-disk by removing \( g \) copies of disjoint 2-disks. As indicated in Figure 6, we denote the boundary components of \( P_g \) by \( \gamma_0, \gamma_2, \ldots, \gamma_{2g} \), and denote some properly embedded arcs of \( P_g \) by \( \gamma_1, \gamma_3, \ldots, \gamma_{2g+1}, \beta_4, \ldots, \beta_{2g-2} \) and \( \beta'_4, \ldots, \beta'_{2g-2} \). On \( \partial(P_g \times [-1,1]) = \Sigma_g \), we define \( c_{2i-1} = \partial(\gamma_{2i-1} \times [-1,1]) \) (\( 1 \leq i \leq g + 1 \)), \( b_{2j} = \partial(\beta_{2j} \times [-1,1]) \), \( b'_{2j} = \partial(\beta'_{2j} \times [-1,1]) \) (\( 2 \leq j \leq g - 1 \)), and \( c_{2k} = \gamma_{2k} \times \{0\} \) (\( 1 \leq k \leq g \)). In Figures 7 and 8 these circles are illustrated and some of them are oriented.

![Figure 7](image-url)

We set a basis of \( H_1(\Sigma_g; \mathbb{Z}) \) as in Figure 9, where \( x_1 = [c_1 \text{ with opposite orientation }] \), \( x_i = [b_{2i} \text{ with proper orientation }] \) (\( 2 \leq i \leq g - 1 \)), \( x_g = [c_{2g+1}] \), and \( y_i = [c_{2i} \text{ with opposite orientation }] \).

![Figure 8](image-url)

![Figure 9](image-url)
A map \( q : H_1(\Sigma_g;\mathbb{Z}_2) \to \mathbb{Z}_2 \) is called a \textit{quadratic form} with respect to the \( \mathbb{Z}_2 \)-homology intersection form \((,)_2 \) on \( \Sigma_g \) (for short, \( \mathbb{Z}_2 \)-\textit{quadratic form} on \( \Sigma_g \)) if \( q(x + y) = q(x) + q(y) + (x,y)_2 \), for each pair of elements \( x, y \) of \( H_1(\Sigma_g;\mathbb{Z}_2) \). For the basis \( \{x_1, y_1, \ldots, x_g, y_g\} \) introduced above, we define \( Arf(q) = \sum_{i=1}^{g} q(x_i)q(y_i) \). We call a \( \mathbb{Z}_2 \)-\textit{quadratic form} \( q \) \textit{even} quadratic form if \( Arf(q) = 0 \) (resp. \( Arf(q) = 1 \)). We define

\[
\mathcal{SP}_g[q] = \{ \psi \in \mathcal{M}_g \mid q(\psi_+(x)) = q(x) \text{ for any } x \in H_1(\Sigma_g;\mathbb{Z}_2) \}.
\]

As is shown in \cite{1}, for two \( \mathbb{Z}_2 \)-\textit{quadratic forms} \( q, q' \) on \( \Sigma_g \), if \( Arf(q) = Arf(q') \), then there is an element \( \psi' \in \mathcal{M}_g \) so that \( q(\psi'_+(x)) = q'(x) \) for any \( x \in H_1(\Sigma_g;\mathbb{Z}_2) \). Therefore, if \( Arf(q) = Arf(q') \), then \( \mathcal{SP}_g[q] \) and \( \mathcal{SP}_g[q'] \) are conjugate in \( \mathcal{M}_g \). By the definition of \( \mathbb{Z}_2 \)-\textit{quadratic form}, values of a quadratic form is completely determined by its value for the basis of \( H_1(\Sigma_g;\mathbb{Z}_2) \). Let \( q_0 \) and \( q_1 \) be \( \mathbb{Z}_2 \)-\textit{quadratic forms} so that \( q_0(x_i) = q_0(y_i) = 0 \) for \( 1 \leq i \leq g \), \( q_1(x_1) = q_1(y_1) = 1 \) and \( q_1(x_j) = q_1(y_j) = 0 \) for \( 2 \leq j \leq g \). Then \( q_0 \) is an even quadratic form and \( q_1 \) an odd quadratic form. If \( q \) is even, then \( \mathcal{SP}_g[q] \) is conjugate to \( \mathcal{SP}_g[q_0] \) in \( \mathcal{M}_g \), on the other hand, if \( q \) is odd, then \( \mathcal{SP}_g[q] \) is conjugate to \( \mathcal{SP}_g[q_1] \) in \( \mathcal{M}_g \). Hence, for the sake of getting some information about groups \( \mathcal{SP}_g[q] \), it suffices to consider only on \( \mathcal{SP}_g[q_0] \) and \( \mathcal{SP}_g[q_1] \). The group \( \mathcal{SP}_g[q_0] \) is called the \textit{even spin mapping class group}, and the group \( \mathcal{SP}_g[q_1] \) is called the \textit{odd spin mapping class group}. The spin mapping class group is defined by Harer \cite{8, 9}. In \cite{11}, we get a system of generators for \( \mathcal{SP}_g[q_0] \). In this section, we will obtain a system of generators for \( \mathcal{SP}_g[q_1] \).

Let \( M \) be a simply connected smooth closed oriented 4-manifold, \((M,F)\) a knotted characteristic surface and \( q_F \) the Rokhlin quadratic form for \((M,F)\). Rokhlin \cite{20} showed (see also \cite{17} and \cite{3}),

\[
Arf(q_F) \equiv \frac{\sigma(M) - F \cdot F}{8} \mod 2,
\]

where \( \sigma(M) \) is the signature of \( M \). By the above formula, we can see \( g_{K_1\#\Sigma_{g-1}} \) is an odd quadratic form. Hence, we get a system of generators for \( \mathcal{SP}_g[q_{K_1\#\Sigma_{g-1}}] \) from that for \( \mathcal{SP}_g[q_1] \).

We introduce some notations used for describing a system of generators for \( \mathcal{SP}_g[q_1] \). For a simple closed curve \( a \) on \( \Sigma_g \), \( T_a \) denotes the Dehn twist about \( a \). The order of composition of maps is the functional one: \( T_bT_a \) means we apply \( T_a \) first, then \( T_b \). For elements \( a, b \) and \( c \) of a group, we write \( \sigma = c^{-1} \),
and $a * b = ab\Phi$. We define some elements of $\mathcal{M}_g$ as follows:

- $C_i = T_{c_i}$, $B_i = T_{b_i}$, $B_i' = T_{b_i}'$,
- $X_i = C_{i+1}C_iC_{i+1}$, $X_i^* = C_{i+1}C_iC_{i+1}$ ($4 \leq i \leq 2g$),
- $Y_{2j} = C_{2j}B_{2j}C_{2j}$, $Y_{2j}^* = C_{2j}B_{2j}C_{2j}$ ($2 \leq j \leq g - 1$),
- $D_i = C_{i}^2$ ($1 \leq i \leq 2g + 1$),
- $DB_{2j} = B_{2j}^2$ ($2 \leq j \leq g - 1$),
- $T_1 = B_4C_5C_7\cdots C_{2g+1}$.

When $g \geq 3$, $G_g$ denotes the subgroup of $\mathcal{M}_g$ generated by $C_1$, $C_2$, $C_3$, $X_i$ ($4 \leq i \leq 2g$), $Y_{2j}$ ($2 \leq j \leq g - 1$), $D_i$ ($1 \leq i \leq 2g + 1$), $DB_{2j}$ ($2 \leq j \leq g - 1$), and $T_1$. It is clear that $X_i^*$ and $Y_{2j}^*$ are elements of $G_g$. When $g = 2$, the subgroup of $\mathcal{M}_2$ generated by $C_1$, $C_2$, $C_3$, $X_4$, and $D_j$ ($1 \leq j \leq 5$) is denoted by $G_2$. For two simple closed curves $l$ and $m$ on $\Sigma_g$, $l$ and $m$ are called $G_g$-equivalent (denoted by $l \sim m$) if there is an element $\phi$ of $G_g$ such that $\phi(l) = m$.

We show that $G_g = SP_g[q_1]$. That is to say, we show,

**Theorem 6.1** If $g = 2$, $SP_2[q_1]$ is generated by $C_1$, $C_2$, $C_3$, $X_4$, and $D_j$ ($1 \leq j \leq 5$). If $g \geq 3$, $SP_g[q_1]$ is generated by $C_1$, $C_2$, $C_3$, $X_i$ ($4 \leq i \leq 2g$), $Y_{2j}$ ($2 \leq j \leq g - 1$), $D_k$ ($1 \leq k \leq 2g + 1$), $DB_{2l}$ ($2 \leq l \leq g - 1$), and $T_1$.

We prove Theorem 6.1 by using the same method as in the proof of Theorem 3.1 in [11]. By an easy calculation, we can check that each generator of $G_g$ is an element of $SP_g[q_1]$, therefore, $G_g \subset SP_g[q_1]$. Hence, we should show $SP_g[q_1] \subset G_g$. In the case where $g = 2$, we use the Reidemeister-Schreier method to show $SP_g[q_1] \subset G_2$ (6.4). In the case where $g \geq 3$, we use other method to show $SP_g[q_1] \subset G_g$. Here, we present this method in outline.

The integral symplectic group is denoted by $Sp(2g, \mathbb{Z})$ and the $\mathbb{Z}_2$ symplectic group by $Sp(2g, \mathbb{Z}_2)$. The generators of these groups are known (on $Sp(2g, \mathbb{Z})$ see for example [12], on $Sp(2g, \mathbb{Z}_2)$ see for example [7 Chap.3]), and these generators are induced by the action of $\mathcal{M}_g$ on $H_1(\Sigma_g, \mathbb{Z})$ or $H_1(\Sigma_g, \mathbb{Z}_2)$. Therefore, the homomorphism $\Phi$: $\mathcal{M}_g \to Sp(2g, \mathbb{Z})$, defined by the action of $\mathcal{M}_g$ on $H_1(\Sigma_g, \mathbb{Z})$, is a surjection, and $\Psi$: $Sp(2g, \mathbb{Z}) \to Sp(2g, \mathbb{Z}_2)$, defined by changing the coefficient from $\mathbb{Z}$ to $\mathbb{Z}_2$, is a surjection. In 6.1, we show $\ker \Phi \subset G_g$. In 6.2 we introduce a finite system of generators for $\ker \Psi$, and, for each generator, we show that one of its inverse by $\Phi$ is an element of $G_g$. Hence, we conclude $\ker \Psi \circ \Phi \subset G_g$. In 6.3 we introduce a finite system of generators

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for $\Psi \circ \Phi(S\mathcal{P}_g[q_1])$, and, for each generator, we show that one of its inverse by $\Psi \circ \Phi$ is an element of $G_g$. As a consequence, we show $S\mathcal{P}_g[q_1] \subset G_g$.

6.1 Step 1 for the case where $g \geq 3$

There is a natural surjection $\Phi: \mathcal{M}_g \to \text{Sp}(2g, \mathbb{Z})$ defined by the action of $\mathcal{M}_g$ on $H_1(\Sigma_g; \mathbb{Z})$. The kernel of $\Phi$ is denoted by $\mathcal{I}_g$ and called the Torelli group. In this subsection, we prove the following lemma:

**Lemma 6.2** The Torelli group $\mathcal{I}_g$ is a subgroup of $G_g$.

![Figure 10](image)

![Figure 11](image)

Johnson [14] showed that, when $g$ is larger than or equal to 3, $\mathcal{I}_g$ is finitely generated. We review his result. For oriented simple closed curves shown in Figure 7, we refer to $(c_1, c_2, \ldots, c_{2g+1})$ and $(c_3, c_5, \ldots, c_{2g+1})$ as *chains*. For oriented simple closed curves $d$ and $e$ which intersect transversely in one point, we construct an oriented simple closed curve $d + e$ from $d \cup e$ as follows: choose a disk neighborhood of the intersection point and in it make a replacement as indicated in Figure 10. For a consecutive subset $\{c_i, c_{i+1}, \ldots, c_j\}$ of a chain, let $c_i + \cdots + c_j$ be the oriented simple closed curve constructed by repeated applications of the above operations. Let $(i_1, \ldots, i_{r+1})$ be a subsequence of
(1, 2, \ldots, 2g + 2) \text{ (resp. } (\beta, 5, \ldots, 2g + 2))\). We construct the union of circles \( C = (c_1 + \cdots + c_{i_2-1}) \cup (c_2 + \cdots + c_{i_3-1}) \cup \cdots \cup (c_{i_r} + \cdots + c_{i_{r+1}-1}) \). If \( r \) is odd, a regular neighborhood of \( C \) is homeomorphic to the compact surface indicated in Figure 11 whose boundaries are \( a \) and \( b \). Let \( \phi = T_b T_a^{-1} \), then \( \phi \) is an element of \( I_g \). We denote \( \phi \) by \([i_1, \ldots, i_{r+1}]\) and call this the odd subchain map of \((c_1, c_2, \ldots, c_{2g+1})\) \text{ (resp. } (c_\beta, c_5, \ldots, c_{2g+1})\) with length \( r + 1 \). Johnson showed the following theorem:

**Theorem 6.3** [14] Main Theorem \text{ For } g \geq 3, \text{ the odd subchain maps of the two chains } (c_1, c_2, \ldots, c_{2g+1}) \text{ and } (c_\beta, c_5, \ldots, c_{2g+1}) \text{ generate } I_g.

We use the following results by Johnson [14].

**Lemma 6.4** [14] \text{ (a) } C_j \text{ commutes with } [i_1, i_2, \ldots] \text{ if and only if } j \text{ and } j + 1 \text{ are either both contained in or are disjoint from the } i \text{'s.}

\text{(b) If } i \neq j + 1, \text{ then } C_j \ast [\cdots, j, i, \cdots] = [\cdots, j + 1, i, \cdots].

\text{(c) If } k \neq j, \text{ then } C_j \ast [\cdots, k, j + 1, \cdots] = [\cdots, k, j, \cdots].

\text{(d) } [1, 2, 3, 4] [1, 2, 3, 4, 6, \ldots, 2n] B_4 \ast [3, 4, 5, \ldots, 2n] = [5, 6, \ldots, 2n] [1, 2, 3, 4, \ldots, 2n], \text{ where } 3 \leq n \leq g.

**Remark 6.5** Johnson showed (d) only in the case where \( n = g \). But we can apply the proof of Lemma 10 of [14] for the case where \( 3 \leq n < g \), since we can regard each surfaces in Figure 18 of [14] as a surface of genus \( n \) which is a submanifold of \( \Sigma_g \).

We prove that any odd subchain map of \((c_1, c_2, c_3, \ldots, c_{2g+1})\) or \((c_\beta, c_5, c_6, \ldots, c_{2g})\) is a product of elements of \( G_g \). The following lemma shows that any odd subchain map of \((c_\beta, c_5, c_6, \ldots, c_{2g})\) is a product of an odd subchain map of \((c_1, c_2, c_3, \ldots, c_{2g+1})\) and elements of \( G_g \).

**Lemma 6.6** For any odd subchain map \( h \) of \((c_\beta, c_5, c_6, \ldots, c_{2g+1})\), there is an element \( g \) of \( G_g \) such that \( g \ast h \) is an odd subchain map of \((c_1, c_2, c_3, \ldots, c_{2g+1})\).

**Proof** If there is not \( \beta \) in the sequence which define \( h \), then \( h \) is an odd subchain map of \((c_1, c_2, c_3, \ldots, c_{2g+1})\). Hence, it suffices to treat the case where the sequence defining \( h \) includes \( \beta \). If \( g = C_{2g+1}^{e_1} \cdots C_5^{e_7} C_5 B^{-1} \) (\( e_i = \pm 1 \)), then, under any choice of signs of \( e_i \), \( g \in G_g \). We can choose signs of \( e_i \) so that \( g \ast h \) is an odd subchain map of \((c_1, c_2, c_3, \ldots, c_{2g+1})\). \( \Box \)

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From here to the end of this subsection, odd subchain maps mean only those of \((c_1, c_2, c_3, \ldots, c_{2g+1})\). The following lemma shows that any odd subchain map, whose length is at least 5 and which begins from 1, 2, 3, 4, 5, is a product of shorter odd subchain maps and elements of \(G_g\).

**Lemma 6.7** For any \(6 \leq n_6 < n_7 < \cdots < n_{2k} \leq 2g + 2\),

\[
(C_4^2) * [1, 2, 3, 5][1, 2, 4, n_6, n_7, \ldots, n_{2k}] (C_4B_4C_4) * [3, 4, 5, n_6, n_7, \ldots, n_{2k}] = [4, n_6, n_7, \ldots, n_{2k}][1, 2, 3, 4, 5, n_6, n_7, \ldots, n_{2k}]
\]

**Proof** By (a) of Lemma 6.4, \(C_4 * [3, 4, 5, \ldots, 2k] = [3, 4, 5, \ldots, 2k]\), and by (d) of Lemma 6.4,

\[
[1, 2, 3, 4][1, 2, 5, 6, \ldots, 2k] \cdot (B_4C_4) * [3, 4, 5, \ldots, 2k] = [5, 6, \ldots, 2k][1, 2, 3, 4, \ldots, 2k].
\]

By applying \(C_4\) to the above equation and remarking that \(C_4 * [1, 2, 3, 4] = (C_4^2) * (C_4 * [1, 2, 3, 4]) = (C_4^2) * [1, 2, 3, 5]\), we get,

\[
(C_4^2) * [1, 2, 3, 5] \cdot [1, 2, 4, 6, \cdots, 2k] \cdot (C_4B_4C_4) * [3, 4, 5, 6, \cdots, 2k] = [4, 6, 7, \cdots, 2k][1, 2, 3, 4, 5, 6, \cdots, 2k].
\]

After proper applications of \(C_6, C_7, \ldots, C_{2g+1}\), we get the equation we need. \(\square\)

**Lemma 6.8** (1) When \(i - k \geq 3\), \((C_{i-1}C_{i-2}C_{i-1}) * [\ldots, k, i, j, \ldots] = [\ldots, k, i - 2, j, \ldots]\).

(2) When \(i - k \geq 2\), \((C_iC_{i-1}C_i) * [\ldots, k, i, i + 1, \ldots] = [\ldots, k, i - 1, i, \ldots]\).

**Proof** Lemma 6.4 shows (1) and (2). \(\square\)

For any odd subchain map \([i_1, i_2, \ldots, i_r]\), we introduce a notation \([[\tau_1, \tau_2, \ldots, \tau_{2g+2}]]: \tau_k = 1\) if \(k\) is a member of \(\{i_1, i_2, \ldots, i_r\}\), and \(\tau_k = 0\) if \(k\) is not a member of \(\{i_1, i_2, \ldots, i_r\}\). For \([[\tau_1, \tau_2, \ldots, \tau_{2g+2}]]: \tau_i (1 \leq i \leq 2g + 2)\) is called the \(i\)-th tack of \([[\tau_1, \tau_2, \ldots, \tau_{2g+2}]\]), and if \(\tau_i = 0\) (resp. 1) then \(\tau_i\) is called a 0-tack (resp. a 1-tack). The number of 1-tacks in \([[\tau_1, \tau_2, \ldots, \tau_{2g+2}]\]) is called the length of \([[\tau_1, \tau_2, \ldots, \tau_{2g+2}]\]). Lemma 6.8 (1) means that, when \(k \geq 3\), if there is a sequence of 0-tacks which begins from the \(k + 1\)-st tack and whose length is at least 2, then the 1-tack subsequent to this 0-tack sequence is moved to left by 2-steps under the action of \(G_g\). Lemma 6.8 (2) means that, when \(k \geq 3\), if there is a sequence of 0-tacks which begins from the \(k + 1\)-st tack and
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We treat the case where the number of 1-tacks in this sequence , 0, ..., 0 is a sequence of 0-tacks. Since $C_1, C_2, C_3 \in G_g$, if there is one 1-tack among $\tau_1, \tau_2, \tau_3$, then $[[\tau_1, \tau_2, \tau_3, \ldots]] \sim G_g ([1, 0, 0, \ldots])$, if there are two 1-tacks among $\tau_1, \tau_2, \tau_3$, then $[[\tau_1, \tau_2, \tau_3, \ldots]] \sim G_g ([1, 1, 0, \ldots])$. The number of 1-tacks in $\tau_1, \tau_2, \tau_3$ is denoted by $h$.

**Lemma 6.9** Any odd subchain map is a product of elements of $G_g$ and the odd subchain maps whose $h$ and $b$ are (1) $h = 3, b = 1$, (2) $h = 3, b = 0$, (3) $h = 2, b = 0$, (4) $h = 1, b = 0$, (5) $h = 0, b = 0$.

**Proof** We treat the case where $h = 3$. If $b \geq 2$, by Lemma 6.7 this odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps.

We treat the case where $h = 2$. If $b \geq 3$,

$$[[1, 1, 0, 1, 1, 1, \ldots]] \xrightarrow{C_3} [[1, 1, 1, 0, 1, 1, \ldots]] \xrightarrow{\text{Lemma 6.7}(2)} [[1, 1, 1, 1, 0, \ldots]],$$

by Lemma 6.7 the last odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps. If $b = 2$,

$$[[1, 1, 0, 1, 0, 1, 0, \ldots]] \xrightarrow{C_3} [[1, 1, 1, 0, 1, 0, \ldots]],$$

the last odd subchain map is in the case where $h = 3, b = 0$. If $b = 1, t$ should be at least 1, and

$$[[1, 1, 0, 1, 0, 1, 0, \ldots]] \xrightarrow{C_3} [[1, 1, 1, 0, 0, 1, 0, \ldots]] \xrightarrow{\text{Lemma 6.7}(1)} [[1, 1, 1, 1, 0, 0, 1, 0, \ldots]],$$

the last odd subchain map is in the case where $h = 3, b = 1$.

We treat the case where $h = 1$. If $b \geq 5$,

$$[[1, 0, 0, 1, 1, 1, 1, \ldots]] \xrightarrow{C_2C_3} [[1, 1, 0, 0, 1, 1, 1, 1, \ldots]] \xrightarrow{\text{Lemma 6.8(2)}} [[1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, \ldots]] \xrightarrow{\text{Lemma 6.8(2)}} [[1, 1, 0, 1, 1, 1, 0, \ldots]] \xrightarrow{C_3} [[1, 1, 1, 0, 1, 1, 1, 0, \ldots]] \xrightarrow{\text{Lemma 6.8(2)}} [[1, 1, 1, 1, 0, 1, 0, \ldots]],$$

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by Lemma 6.7, the last odd subchain map is a product of elements of $G_g$ and the shorter odd subchain maps. If $b = 4$,

$$
[1, 0, 0, 1, 1, 1, 0, \ldots] \rightarrow [1, 1, 0, 0, 1, 1, 1, 0, \ldots] \quad \text{Lemma 6.8(2)}
$$

the last odd subchain map is in the case where $h = 3, b = 0$. If $b = 3$ and $t = 0$,

$$
[1, 0, 0, 1, 1, 0, 0, \ldots] \rightarrow [1, 1, 0, 0, 1, 1, 0, 0, \ldots] \quad \text{Lemma 6.8(2)}
$$

the last odd subchain map is in the case where $h = 3, b = 0$. If $b = 3$ and $t \geq 2$,

$$
[1, 0, 0, 1, 1, 0, 1, 0, \ldots] \rightarrow [1, 1, 0, 0, 1, 1, 0, 1, 0, \ldots] \quad \text{Lemma 6.8(2)}
$$

by Lemma 6.7, the last odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps. If $b = 2$,

$$
[1, 0, 0, 1, 1, 1, 0, \ldots] \rightarrow [1, 1, 0, 0, 1, 1, 0, \ldots] \quad \text{Lemma 6.8(2)}
$$

the last odd subchain map is in the case where $h = 2, b = 0$. If $b = 1$, $t$ should be at least 2,

$$
[1, 0, 0, 1, 0, 0, 1, 0, 1, \ldots] \rightarrow [1, 1, 0, 0, 1, 0, 1, 0, 1, 0, \ldots] \quad \text{Lemma 6.8(1)}
$$

the last odd subchain map is in the case where $h = 2, b = 1$, which we treat before.

We treat the case where $h = 0$. If $b \geq 7$,

$$
[0, 0, 0, 1, 1, 1, 1, 1, 1, \ldots] \rightarrow [1, 0, 0, 0, 1, 1, 1, 1, 1, 1, \ldots] \quad \text{Lemma 6.8(2)}
$$

$$
[1, 0, 0, 1, 1, 1, 1, 1, 1, 0, \ldots] \rightarrow [1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 0, \ldots] \quad \text{Lemma 6.8(2)}
$$

$$
[1, 1, 0, 1, 1, 1, 0, 1, 0, \ldots] \rightarrow [1, 1, 1, 0, 1, 1, 0, 1, 0, \ldots] \quad \text{Lemma 6.8(2)}
$$

$$
[1, 1, 1, 1, 1, 0, 1, 0, \ldots] \rightarrow [1, 1, 1, 1, 1, 0, 1, 0, \ldots] \quad \text{Lemma 6.8(2)}
$$

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by Lemma 6.7, the last odd subchain map is a product of $G_g$ and shorter odd subchain maps. If $b = 6$,

$$[[0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, ...]] \xrightarrow{\text{Lemma 6.8(2)}} [[1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, ...]]$$

$$\xrightarrow{C_1C_2C_3} [[1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, ...]]$$

the last odd subchain map is in the case where $h = 3, b = 0$. If $b = 5$, $t$ should be at least 1 and,

$$[[0, 0, 0, 1, 1, 1, 1, 1, 0, ...]] \xrightarrow{\text{Lemma 6.8(2)}} [[1, 0, 0, 0, 1, 1, 1, 1, 1, 0, ...]]$$

the last odd subchain map is in the case where $h = 3, b = 0$. If $b = 4$, $t = 1$,

$$[[0, 0, 0, 0, 1, 1, 1, 1, 1, 0, ...]] \xrightarrow{\text{Lemma 6.8(2)}} [[1, 0, 0, 0, 1, 1, 1, 1, 1, 0, ...]]$$

the last odd subchain map is in the case where $h = 2, b = 0$. If $b = 3$ and $t = 1$,

$$[[0, 0, 0, 1, 1, 1, 0, 1, 0, ...]] \xrightarrow{\text{Lemma 6.8(2)}} [[1, 0, 0, 0, 1, 1, 0, 1, 0, ...]]$$

the last odd subchain map is in the case where $h = 3, b = 0$. If $b = 3$ and $t \neq 1$, then $t$ should be at least 3 and,

$$[[0, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, ...]] \xrightarrow{\text{Lemma 6.8(2)}} [[1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, ...]]$$
Lemma 6.10  Any odd subchain maps of the 6 cases listed in Lemma 6.4 are products of elements of $G_g$ and odd subchain maps $[[1,1,1,1,0,\ldots,0]]$, $[[1,1,1,0,1,0,\ldots,0]]$, $[[1,1,1,0,1,0,1,0,\ldots,0]]$, and $[[1,1,0,0,1,0,1,0,\ldots,0]]$, where $0,\ldots,0$ are sequences of 0-tacks.

Proof By checking figures of chain maps, for examples $[[1,1,1,1,0,1,0,\ldots,0]]$ and $[[0,0,0,0,1,0,1,0,\ldots,0]]$ indicated in Figure 12 and 13, we see that if an odd subchain map begins from $[[0,0,0,0,\ldots,0]]$ or $[[1,1,1,1,\ldots,0]]$, then this map commutes with $B_4$, hence $B_4$ does not effect on this map.
We treat the case where \( h = 3, b = 1 \). If \( t = 0 \), then this odd subchain map is \([1, 1, 1, 0, 1, 0, \ldots, 0]\). If \( t \neq 0 \), then \( t \) should be at least 2 and,
\[
[[1, 1, 1, 0, 1, 0, 1, 0, 1, 0, \ldots]] \xrightarrow{\mathbb{Z}_3} [[1, 1, 1, 1, 0, 1, 0, \ldots]]],
\]
by Lemma 6.8 the last odd subchain map is a product of elements of \( G_g \) and shorter odd subchain maps.

We treat the case where \( h = 3, b = 0 \). In this case, \( t \) should be an odd integer at least 1. If \( t = 1 \), then this map is \([1, 1, 1, 0, 1, 0, \ldots, 0]\). If \( t = 3 \), then this map is \([1, 1, 1, 0, 1, 0, 1, 0, 1, 0, \ldots, 0]\).

If \( t \geq 5 \),
\[
[[1, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots]] \xrightarrow{\mathbb{Z}_3} [[1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, \ldots]]
\]
by Lemma 6.8.

By the same type of diagrams and subchains, we get the following odd subchain maps:

- \([1, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, \ldots]] \xrightarrow{\mathbb{Z}_2} [[0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, \ldots]]
- \([0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, \ldots]] \xrightarrow{\mathbb{Z}_1} [[0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, \ldots]]
- \([0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, \ldots]] \xrightarrow{\mathbb{Z}_1} [[0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, \ldots]]
- \([0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, \ldots]] \xrightarrow{\mathbb{Z}_1} [[0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, 0, \ldots]]\]
\[ C_1C_2C_3 \rightarrow \begin{bmatrix} [1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 1, \ldots] \end{bmatrix} \]

Lemma 6.8(1) \[ \rightarrow \begin{bmatrix} [1, 0, 0, 0, 1, 1, 1, 1, 1, 0, 0, 1, \ldots] \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} [1, 0, 0, 1, 1, 1, 1, 0, 0, 0, 1, \ldots] \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} [1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, \ldots] \end{bmatrix} \]

Lemma 6.8(2) \[ C_2C_3 \rightarrow \begin{bmatrix} [1, 1, 0, 1, 1, 1, 0, 1, 0, 0, 1, \ldots] \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} [1, 1, 1, 0, 1, 1, 0, 0, 0, 1, \ldots] \end{bmatrix} \]

Lemma 6.8(2) \[ \rightarrow \begin{bmatrix} [1, 1, 1, 1, 0, 1, 0, 0, 0, 1, \ldots] \end{bmatrix} \]

by Lemma 6.7 the last odd subchain map is a product of elements of \( G_g \) and shorter odd subchain maps.

We treat the case where \( h = 2, b = 0 \). In this case, \( t \) should be even integer at least 2. If \( t = 2 \), this map is \([1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 1, \ldots] \). If \( t \geq 4 \),

\[ \begin{bmatrix} [1, 1, 0, 0, 1, 0, 1, 0, 1, 0, 1, \ldots] \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} [1, 0, 0, 1, 1, 1, 0, 1, 0, 1, 0, \ldots] \end{bmatrix} \]

Lemma 6.8(2) \[ \rightarrow \begin{bmatrix} [1, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, \ldots] \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} [0, 0, 0, 1, 1, 1, 1, 1, 0, 1, 0, \ldots] \end{bmatrix} \]

Lemma 6.8(2) \[ \rightarrow \begin{bmatrix} [0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, \ldots] \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} [0, 0, 0, 0, 1, 1, 1, 1, 1, 0, 1, \ldots] \end{bmatrix} \]

Lemma 6.8(2) \[ \rightarrow \begin{bmatrix} [0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, \ldots] \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} [0, 0, 0, 1, 1, 1, 1, 1, 1, 0, 1, \ldots] \end{bmatrix} \]

Lemma 6.8(2) \[ C_1C_2C_3 \rightarrow \begin{bmatrix} [1, 0, 0, 0, 1, 1, 1, 1, 0, 1, 0, \ldots] \end{bmatrix} \]

Lemma 6.8(1) \[ \rightarrow \begin{bmatrix} [1, 0, 0, 1, 1, 1, 1, 0, 1, 0, 0, \ldots] \end{bmatrix} \]

Lemma 6.8(2) \[ \rightarrow \begin{bmatrix} [1, 0, 0, 1, 1, 1, 1, 1, 0, 0, 0, \ldots] \end{bmatrix} \]

Lemma 6.8(1) \[ C_2C_3 \rightarrow \begin{bmatrix} [1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, \ldots] \end{bmatrix} \]

Lemma 6.8(2) \[ \rightarrow \begin{bmatrix} [1, 1, 0, 1, 1, 1, 1, 0, 0, 0, 0, \ldots] \end{bmatrix} \]

\[ \rightarrow \begin{bmatrix} [1, 1, 0, 1, 1, 1, 1, 0, 0, 0, 0, \ldots] \end{bmatrix} \]
We treat the case where \( h = 1, b = 0 \). In this case, \( t \) should be an odd integer at least 3. If \( t = 3 \),

\[
\begin{align*}
&\longrightarrow_{C_3} [[1, 1, 1, 0, 1, 1, 0, 0, 0, 0, \ldots]] \rightarrow_{\text{Lemma 6.8(2)}} [[1, 1, 1, 1, 0, 1, 0, 0, 0, 0, \ldots]], \\
&\longrightarrow_{C_3 C_2 C_1} [[0, 0, 0, 1, 1, 0, 1, 0, 0, 0, \ldots]] \rightarrow_{T_1} [[0, 0, 0, 1, 1, 0, 1, 0, 0, \ldots]], \\
&\longrightarrow_{C_3 C_2 C_1} [[0, 0, 0, 1, 1, 0, 0, 0, 0, \ldots]] \rightarrow_{C_1 C_2 C_3} [[1, 0, 0, 0, 1, 1, 0, 1, 0, 0, \ldots]], \\
&\longrightarrow_{C_3 C_2 C_1} [[1, 0, 0, 0, 1, 1, 0, 0, 0, \ldots]] \rightarrow_{C_3} [[1, 1, 0, 1, 0, 0, 0, 0, 0, \ldots]].
\end{align*}
\]

If \( t \geq 5 \),

\[
\begin{align*}
&\longrightarrow_{C_3 C_2 C_1} [[1, 0, 0, 0, 1, 0, 1, 0, 1, 0, \ldots]] \rightarrow_{\text{Lemma 6.8(2)}} [[0, 0, 0, 1, 1, 0, 0, 1, 0, 0, \ldots]], \\
&\longrightarrow_{C_3 C_2 C_1} [[0, 0, 0, 0, 1, 1, 0, 0, 0, 0, \ldots]] \rightarrow_{T_1} [[0, 0, 0, 1, 1, 0, 0, 1, 0, \ldots]], \\
&\longrightarrow_{C_3 C_2 C_1} [[0, 0, 0, 1, 1, 0, 0, 0, 0, 0, \ldots]] \rightarrow_{C_1 C_2 C_3} [[1, 0, 0, 0, 1, 1, 0, 1, 0, 0, \ldots]], \\
&\longrightarrow_{C_3 C_2 C_1} [[1, 0, 0, 0, 1, 1, 0, 0, 0, 0, \ldots]] \rightarrow_{C_1 C_2 C_3} [[1, 0, 0, 1, 1, 0, 0, 0, 0, \ldots]], \\
&\longrightarrow_{C_3 C_2 C_1} [[1, 0, 0, 1, 1, 0, 0, 0, 0, 0, \ldots]] \rightarrow_{C_3} [[1, 1, 1, 0, 1, 0, 0, 0, 0, \ldots]].
\end{align*}
\]
by Lemma 6.7 the last odd subchain map is a product of elements of $G_g$ and shorter odd subchain maps.

We treat the case where $h = 0$, $b = 0$. In this case, $t$ should be an even integer at least 4. If $t = 4$,

\[
\begin{align*}
\text{Lemma } & \text{6.8(1)} \rightarrow [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, \ldots] \\
\rightarrow & C_3 [1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, \ldots] \\
\text{Lemma } & \text{6.8(2)} \rightarrow [1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, \ldots], \\
\end{align*}
\]

\[
\begin{align*}
\text{Lemma 6.7(1)} \rightarrow & [0, 0, 0, 1, 0, 1, 1, 0, 1, 0, 1, \ldots] \\
\rightarrow & C_1 C_2 C_3 [1, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, \ldots] \\
\text{Lemma 6.7(1)} \rightarrow & [1, 0, 0, 1, 1, 0, 1, 1, 0, 0, 0, 0, \ldots] \\
\rightarrow & C_2 C_3 [1, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, \ldots] \\
\text{Lemma 6.7(1)} \rightarrow & [1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, \ldots] \\
\rightarrow & C_3 [1, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, \ldots] \\
\text{Lemma 6.7(1)} \rightarrow & [1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, \ldots].
\end{align*}
\]

If $t \geq 6$,

\[
\begin{align*}
[0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 0, \ldots] & \rightarrow [0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, \ldots] \\
\rightarrow & \text{(as in the previous case)} \rightarrow [[[1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 1, \ldots]] \\
\rightarrow & \text{Lemma 6.8(1)} [[[1, 1, 1, 1, 0, 1, 0, 1, 0, 1, \ldots]]],
\end{align*}
\]

the last odd subchain map is in the case where $h = 3, b = 1$, which we treat before. \(\square\)

**Lemma 6.11** The odd subchain maps $[[1, 1, 1, 1, 0, \ldots, 0]]$, $[[1, 1, 1, 0, 1, 0, \ldots, 0]]$ and $[[1, 1, 0, 0, 1, 0, 1, 0, \ldots, 0]]$ are elements of $G_g$. 

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Proof In a proof of this Lemma, we use "braid relation", which is explained as follows. Let $a$ and $b$ are simple closed curves on $\Sigma_g$ intersecting transversely in one point, then $T_aT_bT_a^{-1} = T_b^{-1}T_aT_b$, in other word, $T_a * T_b = \overline{T_b} * T_a$.

Let $b'_4$ be the simple closed curve on $\Sigma_g$ indicated in Figure 3 and let $B'_4 = T_{b'_4}$. The odd subchain map $[[1,1,1,0,\ldots,0]]$ is equal to $B_4B'_4$. Since $b'_4 = C_4C_3C_2C_1C_2C_3C_4(b_4)$,

$$B_4B'_4 = B_4C_4C_3C_2C_1C_2C_3C_4C_1C_2C_3C_4C_1C_2C_3C_4C_1,$$

which means that $B_4B'_4$ is a product of squares Dehn twists. By using braid relations of $\mathcal{M}_g$, we can see that these squares of Dehn twists are elements of $G_g$ as follows,

$$(B_4C_4C_3C_2) * (C_1C_1) = (C_1 \cdot C_2 \cdot C_3 \cdot B_4) * (C_4C_4),$$

$$= (C_1 \cdot C_2 \cdot C_3) * (B_4C_4B'_4 \cdot B_4C_4B'_4),$$

$$(B_4C_4C_3) * (C_2C_2) = (C_2 \cdot C_3 \cdot B_4) * (C_4C_4),$$

$$= (C_2 \cdot C_3) * (B_4C_4B'_4 \cdot B_4C_4B'_4),$$

$$(B_4C_4) * (C_3C_3) = (C_3 \cdot B_4) * (C_4C_4) = C_3 * (B_4C_4B'_4 \cdot B_4C_4B'_4),$$

$$B_4 * (C_4C_4) = B_4C_4B'_4 \cdot B_4C_4B'_4,$$

$$((C_4 \cdot C_3 \cdot C_2) * (C_1C_1)) = (C_1 \cdot C_2 \cdot C_3) * (C_4C_4),$$

$$(C_4 \cdot C_3) * (C_2C_2) = (C_2 \cdot C_3) * (C_4C_4),$$

$$C_4 * (C_3C_3) = C_3 * (C_4C_4).$$

Since $C_4 * [[1,1,1,0,\ldots,0]] = [[1,1,1,0,1,0,\ldots,0]],$

$$[[1,1,1,0,1,0,\ldots,0]] = C_4 * (B_4B'_4),$$

$$= (C_4 \cdot B_4C_4C_3C_2) * (C_1C_1) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_2C_2) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_3C_3) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_2C_2) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_3C_3) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_2C_2) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_3C_3) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_2C_2) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_3C_3) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_2C_2) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_3C_3) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_2C_2) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_3C_3) \cdot (C_4 \cdot B_4C_4C_3C_2) \cdot (C_2C_2).$$

This equation shows that $[[1,1,1,0,1,0,\ldots,0]] \in G_g.$
Since \( \overline{C}_4 \overline{C}_3 \overline{C}_6 \overline{C}_5 \overline{C}_4 \ast [[1,1,1,1,0,0,0,1,0,1,0,1,0,0,0,0]] = [[1,1,0,0,0,1,0,1,0,1,0,0,0,0,0]] \),

\[
[[1,1,0,0,0,0,1,0,1,0,0,0,0]] = \overline{C}_4 \overline{C}_3 \overline{C}_6 \overline{C}_5 \overline{C}_4 \ast (B_4 B_4')
\]

\[
= (C_4 C_3 C_6 C_5 C_4 B_4 C_3 C_2) \ast (C_1 C_1) \cdot (C_4 C_3 C_6 C_5 C_4 B_4 C_3 C_2) \ast (C_2 C_2) \cdot
\]

\[
\cdot (C_4 C_3 C_6 C_5 C_4 B_4 C_3) \ast (C_3 C_3) \cdot (C_4 C_3 C_6 C_5 C_4 B_4) \ast (C_4 C_4) \cdot
\]

\[
\cdot (C_4 C_3 C_6 C_5 C_4 C_3 C_2) \ast (C_1 C_1) \cdot (C_4 C_3 C_6 C_5 C_4 C_3 C_3) \ast (C_2 C_2) \cdot
\]

This equation describes [[1,1,0,0,0,0,1,0,1,0,0,0,0,0,0]] as a product of squares of Dehn twists. By using braid relations of \( \cal{M}_g \), we show that these squares of Dehn twists are elements of \( G_g \) as follows,

\[
(C_4 C_3 C_6 C_5 C_4 B_4 C_3 C_2) \ast (C_1 C_1)
\]

\[
= (C_1 \cdot C_4 \overline{B}_4 C_4 \cdot C_2 \cdot C_3 \cdot C_6 C_5 C_6) \ast (C_4 C_4),
\]

\[
(C_4 C_3 C_6 C_5 C_4 B_4 C_4 C_3) \ast (C_2 C_2)
\]

\[
= (C_4 B_4 \overline{B}_4 \cdot C_2 \cdot C_3 \cdot C_6 C_5 C_6) \ast (C_4 C_4),
\]

\[
(C_4 C_3 C_6 C_5 C_4 C_4 B_4 C_4) \ast (C_3 C_3) = (C_4 B_4 \overline{B}_4 \cdot C_3 \cdot C_6 C_5 C_4 \cdot C_6) \ast (C_5 C_5)
\]

\[
= (C_4 B_4 C_4 \cdot C_3 \cdot C_6 C_5 C_4) \ast (C_6 C_5 C_6 \cdot C_6 C_5 C_6),
\]

\[
(C_4 C_3 C_6 C_5 C_4 B_4) \ast (C_4 C_4)
\]

\[
= (C_3 \cdot C_4 B_4 C_4 \cdot C_6 C_5 C_6 \cdot C_6 C_4 C_4) \ast (C_3 C_3),
\]

\[
(C_4 C_3 C_6 C_5 C_4 C_4 C_3 C_2) \ast (C_1 C_1)
\]

\[
= (C_1 \cdot C_3 \cdot C_6 C_5 C_6 \cdot C_6 C_4 \cdot C_4) \ast (C_3 C_3),
\]

\[
(C_4 C_3 C_6 C_5 C_4 C_4 C_3 C_3) \ast (C_2 C_2)
\]

\[
= (C_3 \cdot C_2 \cdot C_6 C_4 \cdot C_6 C_5 C_6 \cdot C_6 C_4 C_4) \ast (C_3 C_3),
\]

\[
(C_4 C_3 C_6 C_5 C_4 C_4 C_4 C_3) \ast (C_4 C_4) = (C_3 \cdot C_4 C_5 C_4) \ast (C_6 C_6).
\]

\[\square\]

**Lemma 6.12** The odd subchain map \([1,1,0,0,1,0,1,0,0,0,0,0,0,0,0,0]\) is an element of \( G_g \).

**Proof** We can show that this odd subchain map is \( G_g \)-equivalent to \([0,0,0,0,0,1,1,1,1,1,1,0,0,0,0,0]\) as follows,

\[
[[1,1,1,0,1,0,1,0,1,0,0,0,0]] \xrightarrow{C_3} [[1,1,0,1,1,0,1,0,1,0,0,0,0,0,0]]
\]
If $g = 4$, $[[0, 0, 0, 0, 1, 1, 1, 1, 1, 1]] = B_4 \overline{B_4} = [[1, 1, 1, 1, 0, 0, 0, 0, 0, 0]]$, which we have already treated in Lemma 6.11. If $g \geq 5$, as we see in Figure 14,

$[[0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 0, \ldots, 0]] = [[1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, \ldots, 1]]$,
in the notation of the last odd subchain map, ... is a sequence of 1-tacks. By Lemma 6.8 (2),

$[[1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 1, \ldots, 1]] \sim_{G_g} [[1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0]]$,

which is a product of elements of $G_g$ and shorter odd subchain maps. □

Therefore, Lemma 6.2 is proved.

### 6.2 Step 2 for the case where $g \geq 3$

Let $\Phi_2$ be the natural homomorphism from $\mathcal{M}_g$ to $\text{Sp}(2g, \mathbb{Z})$ defined by the action of $\mathcal{M}_g$ on the $\mathbb{Z}_2$-coefficient first homology group $H_1(\Sigma_g; \mathbb{Z}_2)$. In this section, we will show the following lemma.

**Lemma 6.13**  $\ker \Phi_2$ is a subgroup of $G_g$.

We denote the kernel of the natural homomorphism from $\text{Sp}(2g, \mathbb{Z})$ to $\text{Sp}(2g, \mathbb{Z}_2)$ by $\text{Sp}^{(2)}(2g)$. We set a basis of $H_1(\Sigma_g; \mathbb{Z})$ as in Figure 0.
and define the intersection form $(,)$ on $H_1(\Sigma_g; \mathbb{Z})$ to satisfy $(x_i, y_j) = \delta_{i,j}$, $(x_i, x_j) = (y_i, y_j) = 0$ $(1 \leq i, j \leq g)$. An element $a$ of $H_1(\Sigma_g; \mathbb{Z})$ is called primitive if there is no element $n(\neq 0, \pm 1)$ of $\mathbb{Z}$, and no element $b$ of $H_1(\Sigma_g; \mathbb{Z})$ such that $a = nb$. For a primitive element $a$ of $H_1(\Sigma_g; \mathbb{Z})$, we define an isomorphism $T_a : H_1(\Sigma_g; \mathbb{Z}) \to H_1(\Sigma_g; \mathbb{Z})$ by $T_a(v) = v + (a, v)a$. This isomorphism is the action of Dehn twist about a simple closed curve representing $a$ on $H_1(\Sigma_g; \mathbb{Z})$. We call $T_a^2$ the square transvection about $a$. Johnson [15] showed the following result.

**Lemma 6.14** $\text{Sp}^{(2)}(2g)$ is generated by square transvections.

In [11], we showed,

**Lemma 6.15** $\text{Sp}^{(2)}(2g)$ is generated by the square transvections about the primitive elements $\sum_{i=1}^{g} (\epsilon_i x_i + \delta_i y_i)$, where $\epsilon_i = 0, 1$ and $\delta_i = 0, 1$.

![Figure 15](image)

For each element $[(\epsilon_1, \delta_1), \ldots, (\epsilon_g, \delta_g)] = \sum_{i=1}^{g} (\epsilon_i x_i + \delta_i y_i)$ (where $\epsilon_i = 0, 1$, $\delta_i = 0, 1$) of $H_1(\Sigma_g; \mathbb{Z})$, we construct an oriented simple closed curve on $\Sigma_g$ which represent this homology class. For each $i$-th block, if $(\epsilon_i, \delta_i) = (0, 0)$, we prepare (0) of Figure 15; if $(\epsilon_i, \delta_i) = (0, 1)$, we prepare (1) of Figure 15; if $(\epsilon_i, \delta_i) = (1, 0)$, we prepare (2) of Figure 15; if $(\epsilon_i, \delta_i) = (1, 1)$, we prepare (3) of Figure 15. After that, we glue them along the boundaries and cap the left boundary component by (-) of Figure 15 and the right boundary component by (+) of Figure 15. We denote this oriented simple closed curve on $\Sigma_g$ by $\{(\epsilon_1, \delta_1), \ldots, (\epsilon_g, \delta_g)\}$. Here, we remark that the action of $T_{[(\epsilon_1, \delta_1), \ldots, (\epsilon_g, \delta_g)]}$ on $H_1(\Sigma_g; \mathbb{Z})$ equals $T_{[(\epsilon_1, \delta_1), \ldots, (\epsilon_g, \delta_g)]}$. And, for any $\phi$ of $\mathcal{M}_g$, $\phi \circ T_{[(\epsilon_1, \delta_1), \ldots, (\epsilon_g, \delta_g)]} \circ \phi^{-1} = T_{\phi([(\epsilon_1, \delta_1), \ldots, (\epsilon_g, \delta_g)])}$.

**Lemma 6.16** For any $\{(\epsilon_1, \delta_1), \ldots, (\epsilon_g, \delta_g)\}$, there is an element $\phi$ of $G_g$ such

---

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that

\[ \phi(\{(\epsilon_1, \delta_1), \cdots, (\epsilon_g, \delta_g)\}) = \{(0,0),(0,1),(0,0),\cdots,(0,0)\} \]

or \[ = \{(0,0),(1,1),(0,0),\cdots,(0,0)\} \]

or \[ = \{(0,0),(0,0),(1,1),\cdots,(0,0)\} \]

or \[ = \{(0,1),(0,0),\cdots,(0,0)\} \]

or \[ = \{(1,1),(0,0),\cdots,(0,0)\} \]

or \[ = \{(0,0),(0,0),\cdots,(0,0)\} \]

Proof If the \( i \)-th block is (3), by the action of \( Y_{2i} \) if \( 2 \leq i \leq g-1 \), \( C_2C_1C_2 \) if \( i = 1 \), and \( C_2gC_{2g+1}C_{2g} \) if \( i = g \), this block is changed to (1). Therefore, it suffices to show this lemma in the case where each block is not (3). First we investigate actions of elements of \( G_g \) on adjacent blocks, say the \( i \)-th block and
the $i+1$-st block, where $i \geq 2$. Each picture of Figure 16 shows the action of $G_g$ on this adjacent blocks.

(a) shows \(\{\bullet \bullet \bullet, (0, 0), (0, 1), \bullet \bullet \bullet\} \sim \{\bullet \bullet \bullet, (0, 1), (0, 0), \bullet \bullet \bullet\}\),

(b) shows \(\{\bullet \bullet \bullet, (0, 0), (1, 1), \bullet \bullet \bullet\} \sim \{\bullet \bullet \bullet, (1, 1), (0, 1), \bullet \bullet \bullet\}\),

(c) shows \(\{\bullet \bullet \bullet, (1, 1), (1, 1), \bullet \bullet \bullet\} \sim \{\bullet \bullet \bullet, (0, 1), (0, 0), \bullet \bullet \bullet\}\),

(d) shows \(\{\bullet \bullet \bullet, (0, 1), (0, 1), \bullet \bullet \bullet\} \sim \{\bullet \bullet \bullet, (0, 1), (0, 0), \bullet \bullet \bullet\}\),

(e) shows \(\{\bullet \bullet \bullet, (0, 1), (1, 1), \bullet \bullet \bullet\} \sim \{\bullet \bullet \bullet, (1, 1), (0, 0), \bullet \bullet \bullet\}\),

where \(\bullet \bullet \bullet\) indicates the part which is not changed by the action of $G_g$. Let $x = \{(\epsilon_1, \delta_1), \cdots, (\epsilon_g, \delta_g)\}$, each of whose block is (0,0) or (0,1) or (1,1). If there are the $j$-th blocks (1, 1) ($j \geq 2$), by (b) and (e), they are gathered to a sequence of (1, 1) blocks which begins from the second block. If there are the $j$-th blocks (0, 1) $(j \geq 2)$, by (a), they are gathered to a sequence of (0, 1) blocks subsequent to the previous sequence of (1, 1) blocks. Hence, we showed,

\[x \sim \{\{\epsilon_1, \delta_1\}, (1, 1), \cdots, (1, 1), (0, 1), \cdots, (0, 1), (0, 0)\}\]
There are 7 cases remained to consider,

In the 8-th case,

\[ \{(\epsilon_1, \delta_1), (0, 0), (0, 0), \cdots, (0, 0), (0, 1), (0, 0), \cdots, (0, 0)\}. \]

or \[ \sim \{(\epsilon_1, \delta_1), (0, 0), (0, 0), \cdots, (0, 0), (0, 1), (0, 0), \cdots, (0, 0)\}. \]

In the second case,

\[ \{(\epsilon_1, \delta_1), (1, 1), (0, 0), \cdots, (0, 0), (0, 1), (0, 0), \cdots, (0, 0)\} \]

\[ \sim \{(\epsilon_1, \delta_1), (1, 1), (0, 1), (0, 0), \cdots, (0, 0)\} \text{(by (a))}. \]

In the 4-th case,

\[ \{(\epsilon_1, \delta_1), (1, 1), (0, 1), (0, 0), \cdots, (0, 0), (0, 1), (0, 0), \cdots, (0, 0)\} \]

\[ \sim \{(\epsilon_1, \delta_1), (1, 1), (0, 1), (0, 1), (0, 0), \cdots, (0, 0)\} \text{(by (a))}. \]

\[ \sim \{(\epsilon_1, \delta_1), (1, 1), (0, 1), (0, 0), \cdots, (0, 0)\} \text{(by (d))}. \]

In the 6-th case,

\[ \{(\epsilon_1, \delta_1), (0, 1), (0, 0), \cdots, (0, 0), (0, 1), (0, 0), \cdots, (0, 0)\} \]

\[ \sim \{(\epsilon_1, \delta_1), (0, 1), (0, 1), (0, 0), \cdots, (0, 0)\} \text{(by (a))}. \]

\[ \sim \{(\epsilon_1, \delta_1), (0, 1), (0, 1), (0, 0), \cdots, (0, 0)\} \text{(by (d))}. \]

In the 8-th case,

\[ \{(\epsilon_1, \delta_1), (0, 0), \cdots, (0, 0), (0, 1), (0, 0), \cdots, (0, 0)\} \]

\[ \sim \{(\epsilon_1, \delta_1), (0, 1), (0, 0), \cdots, (0, 0)\} \text{(by (a))}. \]

Therefore,

\[ x \sim \{(\epsilon_1, \delta_1), (1, 1), (0, 0), (0, 0), \cdots, (0, 0)\}, \]

or \[ \sim \{(\epsilon_1, \delta_1), (1, 1), (0, 0), (0, 0), \cdots, (0, 0)\}, \]

or \[ \sim \{(\epsilon_1, \delta_1), (0, 1), (0, 1), (0, 0), \cdots, (0, 0)\}, \]

or \[ \sim \{(\epsilon_1, \delta_1), (0, 0), (0, 0), (0, 0), \cdots, (0, 0)\}. \]

There are 7 cases remained to consider,

\[ \{(0, 0), (1, 1), (0, 0), (0, 0), \cdots, (0, 0)\}, \quad \{(0, 1), (1, 1), (0, 0), (0, 0), \cdots, (0, 0)\}, \]

\[ \{(0, 1), (1, 1), (0, 1), (0, 0), \cdots, (0, 0)\}, \quad \{(0, 1), (0, 1), (0, 0), (0, 0), \cdots, (0, 0)\}, \]

\[ \{(1, 1), (1, 1), (0, 0), (0, 0), \cdots, (0, 0)\}, \quad \{(1, 1), (1, 1), (0, 1), (0, 0), \cdots, (0, 0)\}, \]

\[ \{(1, 1), (0, 1), (0, 0), (0, 0), \cdots, (0, 0)\}. \]
By (b), the first one is \(G_g\)-equivalent to \(\{(0,0), (0,0), (1,1), (0,0), \ldots, (0,0)\}\). Here, we observe actions of \(G_g\) on the first and the second blocks,

\[
\begin{align*}
\{(0,1), (1,1), \ldots\} &\rightarrow \{(1,1), (1,1), \ldots\} \xrightarrow{C_1} \{(0,0), (0,1), \ldots\}, \\
\{(0,1), (0,1), \ldots\} &\rightarrow \{(1,1), (0,1), \ldots\} \xrightarrow{C_1} \{(0,0), (1,1), \ldots\}.
\end{align*}
\]

By the above observation, we see,

\[
\begin{align*}
\{(0,1), (1,1), (0,0), \ldots, (0,0)\} &\sim_{G_g} \{(1,1), (1,1), (0,0), \ldots, (0,0)\} \\
&\sim_{G_g} \{(0,0), (0,1), (0,0), \ldots, (0,0)\}, \\
\{(0,1), (1,1), (0,1), \ldots, (0,0)\} &\sim_{G_g} \{(1,1), (1,1), (0,1), \ldots, (0,0)\} \\
&\sim_{G_g} \{(0,0), (0,1), (0,1), \ldots, (0,0)\} \\
&\sim_{G_g} \{(0,0), (0,1), (0,0), \ldots, (0,0)\}, \\
\{(0,1), (0,1), (0,0), \ldots, (0,0)\} &\sim_{G_g} \{(1,1), (0,1), (0,0), \ldots, (0,0)\} \\
&\sim_{G_g} \{(0,0), (1,1), (0,0), \ldots, (0,0)\}.
\end{align*}
\]

Hence, we showed that any \(x\) is \(G_g\)-equivalent to the elements listed in the statement of this Lemma.

Since

\[
\begin{align*}
T^{2}_{\{0,1\}, \{0,0\}, \ldots, \{0,0\}} &= D_2, \\
T^{2}_{\{1,1\}, \{0,0\}, \ldots, \{0,0\}} &= (C_1 C_2 C_1^{-1})^2, \\
T^{2}_{\{0,0\}, \{1,1\}, \{0,0\}, \ldots, \{0,0\}} &= (Y_2^*)^2, \\
T^{2}_{\{0,0\}, \{0,1\}, \{0,0\}, \ldots, \{0,0\}} &= D_4, \\
T^{2}_{\{0,0\}, \{0,0\}, \{1,1\}, \{0,0\}, \ldots, \{0,0\}} &= (Y_4^*)^2, \\
T^{2}_{\{0,0\}, \ldots, \{0,0\}} &= id,
\end{align*}
\]

these are elements of \(G_g\). By this fact and Lemma 6.2, Lemma 6.13 is proved.

### 6.3 Step 3 for the case where \(g \geq 3\)

As in the previous subsection, let \(\Phi_2: M_g \rightarrow \text{Sp}(2g, \mathbb{Z}_2)\) be the natural homomorphism. Let \(q_1: H_1(\Sigma_g; \mathbb{Z}_2) \rightarrow \mathbb{Z}_2\) be the quadratic form associated with the intersection form \((,)_2\) of \(H_1(\Sigma_g; \mathbb{Z}_2)\) which satisfies, for the basis \(x_i, y_i\) of \(H_1(\Sigma_g; \mathbb{Z}_2)\) indicated on Figure 9, \(q_1(x_1) = q_1(y_1) = 1\), and \(q_1(x_i) = q_1(y_i) = 0\) when \(i \neq 1\). We define \(O_{q_1}(2g, \mathbb{Z}_2) = \{\phi \in \text{Aut}(H_1(\Sigma_g; \mathbb{Z}_2)) | q_1(\phi(x)) = q_1(x)\} \) for any \(x \in H_1(\Sigma_g; \mathbb{Z}_2)\), then \(\text{SP}_g[q_1] = \Phi_2^{-1}(O_{q_1}(2g, \mathbb{Z}_2))\). Because of Lemma 6.13 if we show \(\Phi_2(G_g) = O_{q_1}(2g, \mathbb{Z}_2)\), then \(G_g = \text{SP}_g[q_1]\) follows.
For any $z \in H_1(\Sigma_g; \mathbb{Z}_2)$ such that $q_1(z) = 1$, we define $T_z(x) = x + (z, x)_2 z$. Then $T_z$ is an element of $O_{q_1}(2g, \mathbb{Z}_2)$, and we call this a $\mathbb{Z}_2$-transvection about $z$. Dieudonné [4] showed the following Theorem (see also [7, Chap.14]).

**Theorem 6.17** [4 Proposition 14 on p.42] When $g \geq 3$, $O_{q_1}(2g, \mathbb{Z}_2)$ is generated by $\mathbb{Z}_2$-transvections.

Let $\Lambda_g$ be the set of $z$ of $H_1(\Sigma_g; \mathbb{Z}_2)$ such that $q(z) = 1$. For any elements $z_1$ and $z_2$ of $\Lambda_g$, we define $z_1 \Box z_2 = z_1 + (z_2, z_1)_2 z_2$. Here, we remark that $T_{z_1}^2 = id$, $T_{z_2}T_{z_1}^{-1}T_{z_2}^{-1} = T_{z_1 \Box z_2}$ and $(z_1 \Box z_2)^{-1} = z_1$. An element $\epsilon_1x_1 + \delta_1y_1 + \cdots + \epsilon_gx_g + \delta_gy_g$ of $H_1(\Sigma_g; \mathbb{Z}_2)$ is denoted by $[(\epsilon_1, \delta_1), \cdots , (\epsilon_g, \delta_g)]$, and each $(\epsilon_i, \delta_i)$ is called the $i$-th block. We remark that $q([(\epsilon_1, \delta_1), \cdots , (\epsilon_g, \delta_g)]) = (\epsilon_1 + \delta_1 + \epsilon_1\delta_1) + \epsilon_2\delta_2 + \cdots \epsilon_g\delta_g$.

**Lemma 6.18** Under the operation $\Box$, $\Lambda_g$ is generated by $x_1, y_1, x_1 + x_2, x_i + y_i$ $(2 \leq i \leq g)$, $x_i + y_i + x_{i+1}$ $(2 \leq i \leq g - 1)$, and $x_i + x_{i+1} + y_{i+1}$ $(2 \leq i \leq g - 1)$.

**Proof** For an element $[(\epsilon_1, \delta_1), \cdots , (\epsilon_g, \delta_g)]$ of $H_1(\Sigma_g; \mathbb{Z}_2)$, let the $j$-th block be the right most block which is $(1, 1)$. When $j \geq 3$, there exist 4 cases of the combination of the $(j - 1)$-st block and the $j$-th block: $[\cdots , (1, 1), (1, 1), \cdots ]$, $[\cdots , (0, 0), (1, 1), \cdots ]$, $[\cdots , (0, 1), (1, 1), \cdots ]$, $[\cdots , (0, 1), (1, 1), \cdots ]$. In each case, we can reduce $j$ at least 1. In fact,

$\begin{align*}
[\cdots , (1, 1), (1, 1), \cdots ] \Box (x_{j-1} + x_j + y_j) &= [\cdots , (0, 1), (0, 0), \cdots ], \\
[\cdots , (0, 0), (1, 1), \cdots ] \Box (x_{j-1} + y_j + x_j) &= [\cdots , (1, 1), (0, 1), \cdots ], \\
[\cdots , (0, 1), (1, 1), \cdots ] \Box (x_{j-1} + x_j + y_j) &= [\cdots , (1, 1), (0, 0), \cdots ], \\
[\cdots , (1, 0), (1, 1), \cdots ] \Box (x_{j-1} + y_j + x_j) &= [\cdots , (1, 1), (0, 0), \cdots ].
\end{align*}$

When $j = 2$, since $q([(\epsilon_1, \delta_1), \cdots , (\epsilon_g, \delta_g)]) = 1$, $[(\epsilon_1, \delta_1), \cdots , (\epsilon_g, \delta_g)]$ must be $[(0, 0), (1, 1), \cdots ]$. Because of an equation

$$
[(0, 0), (1, 1), \cdots ] \Box (x_1 + x_2) \Box y_1 = [(1, 1), (0, 1), \cdots ],
$$

we can reduce $j$ to 1. When $j = 1$, if every $i$-th $(i \geq 2)$ block is $(0, 0)$, then it is $x_1 + y_1$, which is equal to $x_1 \Box y_1$. If there exist at least one of the $i$-th
Corollary 6.19

(i ≥ 2) blocks which are (1,0) or (0,1), then,

\[
\begin{align*}
\cdots \cdots, (0,0), (1,0), \cdots \cdots \cdots \otimes (x_{i-1} + x_i + y_i) &= \cdots \cdots, (1,0), (0,1), \cdots \cdots, \\
\cdots \cdots, (1,0), (0,0), \cdots \cdots \cdots \otimes (x_{i-1} + y_{i-1} + x_i) &= \cdots \cdots, (0,1), (1,0), \cdots \cdots, \\
\cdots \cdots, (0,0), (0,1), \cdots \cdots \cdots \otimes (x_{i-1} + x_i + y_i) &= \cdots \cdots, (1,0), (1,0), \cdots \cdots, \\
\cdots \cdots, (1,0), (0,0), \cdots \cdots \cdots \otimes (x_{i-1} + y_{i-1} + x_i) &= \cdots \cdots, (1,0), (1,0), \cdots \cdots.
\end{align*}
\]

Therefore, we can alter this to an element, each \(i\)-th \((i ≥ 2)\) block of which is (1,0) or (0,1). If the \(i\)-th block of this is (0,1), then

\[
\cdots \cdots, (0,1), \cdots \cdots \cdots \otimes (x_i + y_i) = \cdots \cdots, (1,0), \cdots \cdots.
\]

Therefore, it suffices to consider the case where the first block is (1,1) and other blocks are (1,0). In this case,

\[
\cdots \cdots, (1,0), (1,0) \otimes (x_{g-1} + y_{g-1} + x_g) \cdots \cdots (x_{g-1} + y_{g-1}) = \cdots \cdots, (1,0), (0,0).
\]

By applying the same operation repeatedly, we get \([(1,1), (1,0), (0,0), \cdots, (0,0)],\) which is equal to \(y_1 \otimes (x_1 + x_2)\).

This lemma and Theorem 6.17 shows that

**Corollary 6.19** \(O_{q_1}(2g, \mathbb{Z}_2)\) is generated by \(T_{x_1}, T_{y_1}, T_{x_1+x_2}, T_{x_i+y_i} (2 ≤ i ≤ g), T_{x_i+y_i+x_{i+1}} \cdots \) \((2 ≤ i ≤ g-1), and T_{x_i+x_{i+1}+y_{i+1}} \cdots \) \((2 ≤ i ≤ g-1).\)

Since \(G_g\) is a subgroup of \(\mathcal{SP}_{q_1}[g_1], \Phi_2(G_g) \subset O_{q_1}(2g, \mathbb{Z}_2)\). On the other hand, the fact that \(\Phi_2(C_1) = T_{x_1}, \Phi_2(C_2) = T_{y_1}, \Phi_2(C_3) = T_{x_1+x_2}, \Phi_2(C_4) = T_{x_i+y_i} \cdots \) \((2 ≤ i ≤ g-1), \Phi_2(C_5) = T_{x_i+y_i+x_{i+1}} \cdots \) \((2 ≤ i ≤ g-1), \Phi_2(Y_j) = T_{x_j+y_j} \cdots \) \((2 ≤ j ≤ g-1), \Phi_2(X_{2g}) = T_{x_g+y_g}, \) and Corollary 6.19 show \(\Phi_2(G_g) \subset O_{q_1}(2g, \mathbb{Z}_2)\). Therefore we proved that \(\mathcal{SP}_{g}[q_1] = G_g\) when \(g ≥ 3\).

**6.4 Genus 2 case: Reidemeister-Schreier method**

Birman and Hilden showed the following Theorem.

**Theorem 6.20** \([2]\) \(\mathcal{M}_2\) is generated by \(C_1, C_2, C_3, C_4, C_5\) and its defining relations are:

1. \(C_iC_j = C_jC_i\) if \(|i-j| ≥ 2, i, j = 1, 2, 3, 4, 5,\)
2. \(C_iC_{i+1}C_i = C_{i+1}C_iC_{i+1}, i = 1, 2, 3, 4,\)

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(3) \((C_1C_2C_3C_4C_5)^6 = 1\),
(4) \((C_1C_2C_3C_4C_5C_4C_3C_2C_1)^2 = 1\),
(5) \(C_1C_2C_3C_4C_5C_4C_3C_2C_1 \equiv C_i, \ i = 1, 2, 3, 4, 5\),
where \(\equiv\) means "commute with".

We call (1) (2) of the above relations braid relations. We will use the well-known method, called the Reidemeister–Schreier method [18 §2.3], to show \(\mathcal{S}\mathcal{P}_2[q_1] \subset G_2\). We review (a part of) this method.

Let \(G\) be a group generated by finite elements \(g_1, \ldots, g_m\) and \(H\) be a finite index subgroup of \(G\). For two elements \(a, b\) of \(G\), we write \(a \equiv b \pmod{H}\) if there is an element \(h\) of \(H\) such that \(a = hb\). A finite subset \(S\) of \(G\) is called a coset representative system for \(G \pmod{H}\), if, for each elements \(g\) of \(G\), there is only one element \(\overline{g} \in S\) such that \(g \equiv \overline{g} \pmod{H}\). The set \(\{sg_i\overline{g}_i^{-1} \mid i = 1, \ldots, m, s \in S\}\) generates \(H\).

\[
\begin{array}{cccccc}
[0,1,1,1] & [0,0,1,1] & [1,0,1,1] & [1,1,1,0] & [1,1,0,0] & [1,1,0,1] \\
C_1 & C_2 & C_3 & C_4 & C_5 \\
\end{array}
\]

Figure 17

For the sake of giving a coset representative system for \(\mathcal{M}_2\) modulo \(\mathcal{S}\mathcal{P}_2[q_1]\), we will draw a graph \(\Gamma\) which represents the action of \(\mathcal{M}_2\) on the quadratic forms of \(H_1(\Sigma_2; \mathbb{Z}_2)\) with Arf invariants 1. Let \([\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]\) denote the quadratic form \(q'\) of \(H_1(\Sigma_2; \mathbb{Z}_2)\) such that \(q'(x_1) = \epsilon_1, q'(y_1) = \epsilon_2, q'(x_2) = \epsilon_3, q'(y_2) = \epsilon_4\). Each vertex of \(\Gamma\) corresponds to \(H\) such \(\epsilon\) a coset representative system \(\equiv\) \(\mathcal{M}_2\), we denote its action on \(H_1(\Sigma_2; \mathbb{Z}_2)\) by \((C_i)_s\). For the quadratic form \(q'\) indicated by the symbol \([\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]\), let \(\delta_1 = q'((C_i)_s x_1), \delta_2 = q'((C_i)_s y_1), \delta_3 = q'((C_i)_s x_2), \delta_4 = q'((C_i)_s y_2)\). Then, we connect two vertices, corresponding to \([\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]\), \([\delta_1, \delta_2, \delta_3, \delta_4]\) respectively, by the edge with the letter \(C_i\). We remark that this action is a right action. For simplicity, we omit the edge whose ends are the same vertex. As a result, we get a graph \(\Gamma\) as in Figure 17. The words \(S = \{1, C_5, C_4, C_4C_3, C_4C_3C_2, C_4C_3C_2C_1\}\), which correspond to the edge paths beginning from \([1,1,0,0]\) on \(\Gamma\), define a coset representative system for \(\mathcal{M}_2\) modulo \(\mathcal{S}\mathcal{P}_2[q_1]\). For each element \(g\) of \(\mathcal{M}_2\), we can give a \(\overline{g} \in S\) with using this graph. For example, say \(g = C_2C_4C_5C_3\), we follow an edge path assigned to this word which begins from \([1,1,0,0]\), (note that we read words from left to right) then we arrive at the vertex

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This table shows that \( \text{SP}_2[ q_1 ] \) is a coset representative system \( S \) \( \in G_2 \). Hence, \( C_4 C_3 \) \( \in \text{SP}_2[ q_1 ] \). We list in Table 1 the set of generators \( \{ sC_i sC_i^{-1} \mid i = 1, \ldots, 5, \ s \in S \} \) of \( \text{SP}_2[ q_1 ] \). In Table 1, the vertical direction is a coset representative system \( S \), horizontal direction is a set of generators \( \{ C_1, C_2, C_3, C_4, C_5 \} \). We can check this table by Figure 17 and braid relations. For example,

\[
C_4 C_3 C_2 C_1 \cdot C_2 C_3 C_2 C_1 \cdot C_2^{-1} = C_4 C_3 C_2 C_1 C_2 (C_4 C_3 C_2 C_1)^{-1}
\]

Then we can see \( \text{SP}_2[ q_1 ] \) \( \subset G_2 \).

This table shows that \( \text{SP}_2[ q_1 ] \subset G_2 \).

### 7 Proof of Theorem 5.1

We embed \( H_{g-1} \) standardly in \( S^3 = \partial D_4 \) such that there is a 2-sphere separating \( F_{3,3} \) and \( H_{g-1} \), and make a connected sum \( F_{3,3} \# \partial H_{g-1} \) as indicated in Figure 18. Then, we can see \( (\mathbb{C}P^2, K_3 \# \Sigma_{g-1}) = (\mathbb{C}P^2, (F_{3,3} \# \partial H_{g-1}) \cup D_3) \), where \( K_3 \) is the non-singular plane curve of degree 3 and \( D_3 \) is parallel three disks which is used to construct \( K_3 \) in Figure 18. We identify \( K_3 \# \Sigma_{g-1} \) with \( \Sigma_{g} \) so that
simple closed curves with the same symbol are identified. Then \( q_{K_3#\Sigma_{g-1}} = q_1 \).

We will show that each element of \( S\mathcal{P}_g[q_{K_3#\Sigma_{g-1}}] = S\mathcal{P}_g[q_1] \) is extendable.

Each regular neighborhood of \( c_1, c_2, c_3, C_{i+1}(c_i) \) (4 \( \leq i \leq 2g \)), and \( C_{2j}(b_{2j}) \) (2 \( \leq j \leq g-1 \)) is Hopf band. Therefore, by Proposition 2.1, \( C_1, C_2, C_3, C_{i+1}C_iC_{i+1} \) (4 \( \leq i \leq 2g \)), and \( C_{2j}B_{2j}C_{2j} \) (2 \( \leq j \leq g-1 \)) are elements of \( E(\mathbb{CP}^2, K_3#\Sigma_{g-1}) \). Each regular neighborhood of \( c_i \) (4 \( \leq i \leq 2g+1 \)), \( b_{2j} \) (2 \( \leq i \leq g-1 \)) is an annulus standardly embedded in \( S^3 = \partial D^4 \). We can deform this annulus as indicated in Figure 1. Therefore, \( C_i^2 \) (4 \( \leq i \leq 2g+1 \)), \( B_{2j}^2 \) (2 \( \leq j \leq g-1 \)) are elements of \( E(\mathbb{CP}^2, K_3#\Sigma_{g-1}) \). Finally, the extendability of \( B_4C_5C_7...C_{2g+1} \) follows from the proof of Lemma 2.2 in [11]. Therefore, we showed \( S\mathcal{P}_g[q_{K_3#\Sigma_{g-1}}] \subset E(\mathbb{CP}^2, K_3#\Sigma_{g-1}) \). On the other hand, by the definition of the Rokhlin quadratic form \( q_{K_3#\Sigma_{g-1}} \), we see \( E(\mathbb{CP}^2, K_3#\Sigma_{g-1}) \subset S\mathcal{P}_g[q_{K_3#\Sigma_{g-1}}] \). Theorem 5.1 follows.

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