Abstract  Which slopes can or cannot appear as Seifert fibered slopes for
hyperbolic knots in the 3-sphere $S^3$? It is conjectured that if $r$-surgery on
a hyperbolic knot in $S^3$ yields a Seifert fiber space, then $r$ is an integer.
We show that for each integer $n \in \mathbb{Z}$, there exists a tunnel number one,
hyperbolic knot $K_n$ in $S^3$ such that $n$-surgery on $K_n$ produces a small
Seifert fiber space.

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slopes

This paper is dedicated to Donald M. Davis on the occasion of his 60th
birthday.

1 Introduction

Let $K$ be a knot in the 3-sphere $S^3$ with a tubular neighborhood $N(K)$. Then
the set of slopes for $K$ (i.e., $\partial N(K)$-isotopy classes of simple loops on $\partial N(K)$)
is identified with $\mathbb{Q} \cup \{\infty\}$ using preferred meridian-longitude pair so that a
meridian corresponds to $\infty$. A slope $\gamma$ is said to be integral if a representative
of $\gamma$ intersects a meridian exactly once, in other words, $\gamma$ corresponds to an
integer under the above identification. In the following, we denote by $(K;\gamma)$
the 3-manifold obtained from $S^3$ by Dehn surgery on a knot $K$ with slope $\gamma$,
i.e., by attaching a solid torus to $S^3-\text{int}N(K)$ in such a way that $\gamma$ bounds a
meridian disk of the filled solid torus. If $\gamma$ corresponds to $r \in \mathbb{Q} \cup \{\infty\}$, then
we identify $\gamma$ and $r$ and write $(K;r)$ for $(K;\gamma)$.

We denote by $\mathcal{L}$ the set of lens slopes $\{r \in \mathbb{Q} | \exists$ hyperbolic knot $K \subset S^3$
such that $(K;r)$ is a lens space $\}$, where $S^3$ and $S^2 \times S^1$ are also considered
as lens spaces. Then the cyclic surgery theorem [7] implies that $\mathcal{L} \subset \mathbb{Z}$. A
result of Gabai [10, Corollary 8.3] shows that $0 \notin \mathcal{L}$, a result of Gordon and Luecke [14] shows that $\pm 1 \notin \mathcal{L}$. Furthermore, a result of Kronheimer, Mrowka, Ozsváth and Szabó [20] implies that $\pm 3, \pm 4 \notin \mathcal{L}$. Besides, Berge [4, Table of Lens Spaces] suggests that if $n \in \mathcal{L}$, then $|n| \geq 18$ and not every integer $n$ with $|n| \geq 18$ appears in $\mathcal{L}$. Fintushel and Stern [9] had shown that 18-surgery on the $(-2,3,7)$ pretzel knot yields a lens space.

Which slope (rational number) can or cannot appear in the set of Seifert fibered slopes $\mathcal{S} = \{ r \in \mathbb{Q} \mid \exists$ hyperbolic knot $K \subset S^3$ such that $(K;r)$ is Seifert fibered$\}$? It is conjectured that $\mathcal{S} \subset \mathbb{Z}$ [12].

The purpose of this paper is to prove:

**Theorem 1.1** For each integer $n \in \mathbb{Z}$, there exists a tunnel number one, hyperbolic knot $K_n$ in $S^3$ such that $(K_n;n)$ is a small Seifert fiber space (i.e., a Seifert fiber space over $S^2$ with exactly three exceptional fibers).

**Remark** Since $K_n$ has tunnel number one, it is embedded in a genus two Heegaard surface of $S^3$ and strongly invertible [26, Lemma 5]. See [22, Question 3.1].

Theorem 1.1 together with the previous known results, shows:

**Corollary 1.2** $\mathcal{L} \subset \mathbb{Z} \subset \mathcal{S}$.

**Remarks**

(1) For the set of reducing slopes $\mathcal{R} = \{ r \in \mathbb{Q} \mid \exists$ hyperbolic knot $K \subset S^3$ such that $(K;r)$ is reducible$\}$, Gordon and Luecke [13] have shown that $\mathcal{R} \subset \mathbb{Z}$. In fact, the cabling conjecture [11] asserts that $\mathcal{R} = \emptyset$.

(2) For the set of toroidal slopes $\mathcal{T} = \{ r \in \mathbb{Q} \mid \exists$ hyperbolic knot $K \subset S^3$ such that $(K;r)$ is toroidal$\}$, Gordon and Luecke [15] have shown that $\mathcal{T} \subset \mathbb{Z}/2$ (integers or half integers). In [28], Teragaito shows that $\mathbb{Z} \subset \mathcal{T}$ and conjectures that $\mathcal{T} \not\subset \mathbb{Z}/2$.

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2 Hyperbolic knots with Seifert fibered surgeries

Our construction is based on an example of a longitudinal Seifert fibered surgery given in [17].

Let $k \cup c$ be a 2-bridge link given in Figure 1 and let $K_n$ be a knot obtained from $k$ by $\frac{1}{-n+4}$-surgery along $c$.

![Figure 1: $K_n$](image)

We shall say that a Seifert fiber space is of type $S^2(n_1, n_2, n_3)$ if it has a Seifert fibration over $S^2$ with three exceptional fibers of indices $n_1, n_2$ and $n_3$ ($n_i \geq 2$). Since $K_4$ is unknotted, $(K_4; 4)$ is a lens space $L(4, 1)$. For the other $n$’s, we have:

**Lemma 2.1** $(K_n; n)$ is a small Seifert fiber space of type $S^2(3, 5, |4n - 15|)$ for any integer $n \neq 4$.

**Proof** Since the linking number of $k$ and $c$ is one (with suitable orientations), $(K_n; n)$ has surgery descriptions as in Figure 2.

![Figure 2: Surgery descriptions of $(K_n; n)$](image)

Let us take the quotient by the strong inversion of $S^3$ with an axis $L$ as shown in Figure 3.

Then we obtain a branch knot $b'$ which is the image of the axis $L$. The Montesinos trick ([25], [6]) shows that $-\frac{1}{2}, -1, \frac{3n-11}{-n+4}$ and 1-surgery on $t_1, t_2, c$ and $d$.
$k$ in the upstairs correspond to $-\frac{1}{2}, -1, \frac{3n-11}{-n+4}$ and 1-untangle surgery on $b'$ in the downstairs, where an $r$-untangle surgery is a replacement of $\frac{1}{r}$-untangle by $r$-untangle. (We adopt Bleiler’s convention [5] on the parametrization of rational tangles.) These untangle surgeries convert $b'$ into a link $b$ (Figure 3).

Following the sequence of isotopies in Figures 3 and 4 we obtain a Montesinos link $M(\frac{2}{5}, -\frac{2}{5}, \frac{n-4}{3n-19})$.

Since $(K_n; n)$ is the double branched cover of $S^3$ branched over the Montesinos
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-1

\[
\begin{array}{c}
\frac{1}{2} \\
3n - 11 \\
n - 4
\end{array}
\]

-1

\[
\begin{array}{c}
\frac{1}{2} \\
3n - 11 \\
n - 4
\end{array}
\]

\[
\begin{array}{c}
-\frac{5}{2} \\
\frac{3}{2} \\
4n - 15 \\
n - 4
\end{array}
\]

\[
\begin{array}{c}
2 \\
5 \\
n - 4
\end{array}
\]

\[
\begin{array}{c}
-2 \\
3 \\
4n - 15
\end{array}
\]

Figure 4: Continued from Figure 3

link \( M\left(\frac{2}{5} - \frac{2}{5}, \frac{n - 4}{4n - 15}\right) \), \((K_n; n)\) is a Seifert fiber space of type \(S^2(3, 5, |4n - 15|)\) as desired.

Lemma 2.2 The knot \(K_n\) is hyperbolic if \(n \neq 3, 4, 5\).

Proof Note that the 2-bridge link given in Figure 1 is not a \((2, p)\)-torus link, and hence by [23] it is a hyperbolic link. If \(n \neq 3, 4, 5\), then \(1 - n + 4 > 1\) and it follows from [11, Theorem 1] (also [14, Theorem 1.2]) that \(K_n\) is a hyperbolic knot. See also [19, Corollary A.2], [21, Theorem 1.2] and [2, Theorem 1.1].
Remark It follows from [21], [18] that $K_n$ is a nontrivial knot except when $n = 4$. An experiment using Weeks’ computer program “SnapPea” [31] suggests that $K_3$ and $K_5$ are hyperbolic, but we will not use this experimental results.

Lemma 2.3 The knot $K_n$ has tunnel number one for any integer $n \neq 4$.

Proof Since the link $k \cup c$ is a two-bridge link, the tunnel number of $k \cup c$ is one with unknotting tunnel $\tau$; A regular neighborhood $N(k \cup c \cup \tau)$ is a genus two handlebody and $S^3 - \text{int}N(k \cup c \cup \tau)$ is also a genus two handlebody, see Figure 5.

Then the general fact below (in which $k \cup c$ is not necessarily a two-bridge link) shows that the tunnel number of $K_n$ is less than or equal to one. Since our knot $K_n \ (n \neq 4)$ is knotted in $S^3$, the tunnel number of $K_n$ is one.

Claim 2.4 Let $k \cup c$ be a two component link in $S^3$ which has tunnel number one. Assume that $c$ is unknotted in $S^3$. Then every knot obtained from $k$ by twisting along $c$ has tunnel number at most one.

Proof Let $\tau$ be an unknotting tunnel and $V$ a regular neighborhood of $k \cup c \cup \tau$ in $S^3$; $V$ is a genus two handlebody. Since $\tau$ is an unknotting tunnel for $k \cup c$, by definition, $W = S^3 - \text{int}V$ is also a genus two handlebody. Take a small tubular neighborhood $N(c) \subset \text{int}V$ and perform $-\frac{1}{n}$-surgery on $c$ using $N(c)$. Then we obtain a knot $k_n$ as the image of $k$ and obtain a genus two handlebody $V(c; -\frac{1}{n})$. Note that $V(c; -\frac{1}{n})$ and $W$ define a genus two Heegaard splitting of $S^3$, see Figure 6 where $c^*_n$ denotes the core of the filled solid torus.

Then it is easy to see that an arc $\tau_n$ given by Figure 6 is an unknotting tunnel for $k_n$ as desired.

Now we are ready to prove Theorem 1.1. Lemmas 2.1, 2.2 and 2.3 show that our knots $K_n$ enjoy the required properties, except for $n = 3, 4, 5$. To prove
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Theorem 1.1, we find hyperbolic knots $K'_n$ so that $(K'_n; n)$ is Seifert fibered for $n = 3, 4, 5$ (instead of showing that $K_3$, $K_5$ are hyperbolic). As the simplest way, let $K'_3$, $K'_4$ and $K'_5$ be the mirror image of $K'_3$, $K'_4$ and $K'_5$, respectively. Since $K_3$, $K_4$ and $K_5$ are tunnel number one, hyperbolic knots by Lemmas 2.2 and 2.3 their mirror images $K'_3$, $K'_4$ and $K'_5$ are also tunnel number one, hyperbolic knots. It is easy to observe that $(K'_3; 3)$ (resp. $(K'_4; 4)$, $(K'_5; 5)$) is the mirror image of $(K_3; -3)$ (resp. $(K_4; -4)$, $(K_5; -5)$). By Lemma 2.1, $(K_3; -3)$, $(K_4; -4)$ and $(K_5; -5)$ are Seifert fibered, and hence $(K'_3; 3)$, $(K'_4; 4)$ and $(K'_5; 5)$ are also Seifert fibered. Putting $K_n$ as $K'_n$ for $n = 3, 4, 5$, we finish a proof of Theorem 1.1.

3 Identifying exceptional fibers

In [24], Miyazaki and Motegi conjectured that if $K$ admits a Seifert fibered surgery, then there is a trivial knot $c \subset S^3$ disjoint from $K$ which becomes a Seifert fiber in the resulting Seifert fiber space, and verified the conjecture for several Seifert fibered surgeries [24, Section 6], see also [8]. Furthermore, computer experiments via “SnapPea” [31] suggest that such a knot $c$ is realized by a short closed geodesic in the hyperbolic manifold $S^3 - K$, for details see [24, Section 9], [27].

In this section, we verify the conjecture for Seifert fibered surgeries given in Theorem 1.1.

Recall that $K_n$ is obtained from $k$ by $\frac{1}{n+4}$-surgery on the trivial knot $c$ (i.e., $(n-4)$-twist along $c$), see Figure 1. Denote by $c_n$ the core of the filled solid torus. Then $K_n \cup c_n$ is a link in $S^3$ such that $c_n$ is a trivial knot.

**Lemma 3.1** After $n$-surgery on $K_n$, $c_n$ becomes an exceptional fiber of index $\lfloor 4n - 15 \rfloor$ in the resulting Seifert fiber space $(K_n; n)$. 

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Proof Following the sequences given by Figures 3 and 4, we have a Montesinos link with three arcs $\gamma$, $\tau_1$ and $\tau_2$ as in Figure 7, where $n = 1$ in the final Montesinos link, and $\gamma$, $\tau_1$, $\tau_2$ and $\kappa$ are the images of $c$, $t_1$, $t_2$ and $k$, respectively.

From Figure 7 we recognize that $t_1$, $t_2$ and $c$ become exceptional fibers of indices 5, 3 and $|4n - 15|$, respectively in $(K_n; n)$.

For $n \neq 3, 4, 5$, $c_n$ becomes an exceptional fiber of index $|4n - 15|$, which is the unique maximal index, in $(K_n; n)$. Experiments via “SnapPea” [31] suggest that $c_n$ is a shortest closed geodesic in $S^3 - K_n$ ($n \neq 3, 4, 5$). For sufficiently large $|n|$, hyperbolic Dehn surgery theorem [29], [30] shows that $c_n$ is the unique shortest closed geodesic in $S^3 - K_n$.

Let us assume that $n = 3, 4, 5$. Then we have put $K_n$ as the mirror image of $K_{-n}$ in the proof of Theorem 1.1. Let $k' \cup c'$ be the mirror image of the link $k \cup c$. Then $K_n$ is obtained also from $k'$ by $\frac{3n-11}{-n+4}$-surgery on $c'$ (i.e., $(n + 4)$-twist along $c'$); we denote the core of the filled solid torus by $c'_n$. Note that there is an orientation reversing diffeomorphism from $(K_{-n}; -n)$ to $(K_n; n)$ sending $c_{-n}$ (regarded as a fiber in $(K_{-n}; -n)$) to $c'_n$ (regarded as a fiber in $(K_n; n)$). Thus the above observation implies that $c'_n$ becomes an exceptional fiber of index $|4n + 15|$, which is the unique maximal index, in $(K_n; n)$ ($n = 3, 4, 5$).
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References


[19] P Kronheimer, T Mrowka; Dehn surgery, the fundamental group and SU(2). arXiv:math.GT/0312322


[31] J Weeks; SnapPea: a computer program for creating and studying hyperbolic 3-manifolds, freely available from http://geometrygames.org/SnapPea/

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