The topological Hawaiian earring group does not embed in the inverse limit of free groups

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Abstract Endowed with natural topologies, the fundamental group of the Hawaiian earring continuously injects into the inverse limit of free groups. This note shows the injection fails to have a continuous inverse. Such a phenomenon was unexpected and appears to contradict results of another author.

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1 Introduction

Quite generally the based fundamental group $\pi_1(X,p)$ of a space $X$ becomes a topological group whose topology is invariant under the homotopy type of the underlying space $X$ (Corollary 3.4 [1]). In the context of spaces complicated on the small scale the utility of this invariant is emerging. For example topological $\pi_1$ has the potential to distinguish spaces when the algebraic homotopy groups fail to do so [2]. Unfortunately even in the simplest cases the topological properties of $\pi_1(X,p)$ can be challenging to understand.

Consider the familiar Hawaiian earring $X = \bigcup_{n=1}^{\infty} S_n$, (the union of a null sequence of simple closed curves $S_n$ joined at a common point) and the canonical homomorphism $\phi: \pi_1(X) \to \lim_\leftarrow \pi_1(\bigcup_{i=1}^{n} S_i)$.

The paper [1, page 370] seems to claim that $\phi$ is also a homeomorphism onto its image ("$\phi^{-1}$ is surely continuous as well..."). The intent of this note is to show that such a claim is false. The monomorphism $\phi$ is not a homeomorphism onto its image, and thus $\phi$ fails to be a topological embedding (Theorem 2.1).

To prove this we consider the sequence $[(y_1 \ast y_n \ast y_1^{-1} \ast y_n^{-1})^n]$ where $y_i$ loops counterclockwise around the $i$th circle. The sequence diverges in $\pi_1(X,p)$ with the quotient topology but the sequence converges to the trivial element in the inverse limit space $\lim_\leftarrow \pi_1(\bigcup_{i=1}^{n} S_i)$.
2 Main Result

Suppose $X$ is a topological space and $p \in X$. Endowed with the compact open topology, let $C_p(X) = \{f : [0,1] \to X \suchthat f$ is continuous and $f(0) = f(1) = p\}$. Then the topological fundamental group $\pi_1(X,p)$ is the quotient space of $C_p(X)$ obtained by treating the path components of $C_p(X)$ as points. Thus, letting $q : C_p([0,1],X) \to \pi_1(X,p)$ denote the canonical surjection, a set $A \subset \pi_1(X,p)$ is closed in $\pi_1(X,p)$ if and only if $q^{-1}(A)$ is closed in $C_p([0,1],X)$.

The space $Y$ is said to be $T_1$ if the one point subsets of $Y$ are closed.

If $A_1, A_2$, are topological spaces and $f_n : A_{n+1} \to A_n$ is a continuous surjection then, (endowing $A_1 \times A_2...$ with the product topology) the inverse limit space $\lim_\leftarrow A_n = \{(a_1,a_2,...) \in (A_1 \times A_2...) \mid f_n(a_{n+1}) = a_n\}$.

The map $f : [0,1] \to Y$ is of the form $\alpha_1 \ast \alpha_2... \ast \alpha_n$ if there exists a partition $t_0 \leq t_1... \leq t_n$ of $[0,1]$ such that for each $i \geq 1$ we have $f_{[t_{i-1},t_i]} = \alpha_i$.

For the remainder of the paper we use the following notation.

Let $X_n = \bigcup_{i=1}^n \{(x,y) \in R^2 \mid (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$. Note since $X_n$ is locally contractible the path components of $C_p(X_n)$ are open in $C_p(X_n)$ and hence the topological group $\pi_1(X_n,p)$ has the discrete topology.

Let $r_n^\ast : \pi_1(X_n,p) \to \pi_1(X_{n-1},p)$ denote the epimorphism induced by the retraction $r_n : X_n \to X_{n-1}$ collapsing the $n^{th}$ circle to the point $p = (0,0)$. Let $\lim_\leftarrow \pi_1(X_n,p)$ denote the inverse limit space under the maps $r_n^\ast$.

Let $X = \bigcup_{n=1}^\infty X_n$ denote the familiar Hawaiian and let $R_n : X \to X_n$ denote the retraction fixing $X_n$ pointwise and collapsing $\bigcup_{i=n+1}^\infty X_i$ to the point $p$.

The formula $\phi([f]) = ([R_1(f)],[R_2(f)],...) $ determines a canonical homomorphism $\phi : \pi_1(X,p) \to \lim_\leftarrow \pi_1(X_n,p)$.

**Remark** The homomorphism $\phi : \pi_1(X,p) \to \lim_\leftarrow \pi_1(X_n,p)$ is continuous (Proposition 3.3 [1]) and one to one (Theorem 4.1 [3]). Since $\pi_1(X_n,p)$ is discrete the space $\Pi_{n=1}^\infty \pi_1(X_n,p)$ is metrizable and in particular the subspace $\lim_\leftarrow \pi_1(X_n,p)$ is a $T_1$ space. Consequently $\pi_1(X,p)$ is a $T_1$ space since $\phi$ is continuous and one to one. Thus the path components of $C_p(X)$ are closed in $C_p(X)$.

**Theorem 2.1** The injection $\phi : \pi_1(X,\{p\}) \hookrightarrow \lim_\leftarrow \pi_1(X_n,p)$ is not a topological embedding.
The topological Hawaiian earring group

Proof Let \( q = (2,0) \) in \( X_1 \). For a loop \( f : [0,1] \to \bigcup_{i=1}^\infty X_i \) with base point \( p = (0,0) \) define the oscillation number \( O_q(f) \) as the maximal \( n \) such that there exist \( 0 = t_0 < t_1 \cdots t_{2n-1} < t_{2n} = 1 \) with \( f(t_{2i}) = p \) and \( f(t_{2i+1}) = q \). Let \( y_i \in C_p(X) \) loop once counterclockwise around the \( i \)th circle and let \( y_i^{-1} \in C_p(X) \) loop once clockwise around the \( i \)th circle.

First note that if \( f \in C_p(\bigcup_{i=1}^\infty X_i) \) is path homotopic to a map of the form 
\[ (y_1^{-1} * y_n^{-1} * y_1 * y_n)^n \]
then \( O_q(f) \geq 2n \) for \( n \geq 2 \). To see this first observe \( O_q(f) = O_q(R_n f) \). Now recall \( \pi_1(X_n,p) \) is canonically isomorphic to the free group on generators \( \{y_1,...y_n\} \). Thus if \( w \) is an (unreduced) word corresponding to \( R_n f \) then each step of the algebraic reduction of \( w \) to \( (y_1^{-1} y_n^{-1} y_1 y_n)^n \) never raises the oscillation number of the corresponding path in \( X_n \). Hence \( O_q(f) \geq O_q((y_1^{-1} y_n^{-1} * y_1 * y_n)^n) = 2n \).

To prove \( \phi \) is not an embedding consider the set \( A \subset \pi_1(X,p) \) defined as
\[ A = \{[f_2], [f_3],...\} \]
where \( f_n \) is of the form \( (y_1^{-1} * y_n^{-1} * y_1 * y_n)^n \). To see that \( A \) is closed in \( \pi_1(X,p) \) consider the union of (closed) path components \( B = \bigcup_{n=2}^\infty [f_n] \subset C_p(X) \). Observe if \( f \in C_p(X) \) there exists an open neighborhood \( U \subset C_p(X) \) such that \( O_q(f) \geq O_q(g) \) for each \( g \in U \). Thus \( U \cap [f_n] \neq \emptyset \) for at most finitely many of the closed sets \( [f_n] \). Hence \( B \) is closed in \( C_p(X) \) and consequently \( A \) is closed in \( \pi_1(X,p) \). On the other hand \( \phi(A) \) is not closed in the image of \( \phi \) since the sequence \( \{\phi([f_n])\} \) converges to the trivial element in \( \lim_{\to} \pi_1(X_n,p) \). Hence \( \phi \) is not a homeomorphism from \( \pi_1(X,p) \) onto the image of \( \phi \).

References

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