On diffeomorphisms over surfaces trivially embedded in the 4-sphere

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Abstract  A surface in the 4-sphere is trivially embedded, if it bounds a 3-dimensional handle body in the 4-sphere. For a surface trivially embedded in the 4-sphere, a diffeomorphism over this surface is extensible if and only if this preserves the Rokhlin quadratic form of this embedded surface.

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1 Introduction

We denote the closed oriented surface of genus \( g \) by \( g \), the mapping class group of \( g \) by \( \Gamma_g \). Let \( g! : \mathbb{S}^4 \to \mathbb{S}^4 \) be an embedding, and \( K \) be its image. We call \((\mathbb{S}^4; K)\) a \( g \)-knot. Two \( g \)-knots \((\mathbb{S}^4; K)\) and \((\mathbb{S}^4; K')\) are equivalent if there is a diffeomorphism of \( \mathbb{S}^4 \) which brings \( K \) to \( K' \). A 3-dimensional handlebody \( H_g \) is an oriented 3-manifold which is constructed from a 3-ball with attaching \( g \) 1-handles. Any embeddings of \( H_g \) into \( \mathbb{S}^4 \) are isotopic each other. Therefore, \((\mathbb{S}^4; \partial H_g)\) is unique up to equivalence. We call this \( g \)-knot \((\mathbb{S}^4; \partial H_g)\) a trivial \( g \)-knot and denote this by \((\mathbb{S}^4; g)\). For a \( g \)-knot \((\mathbb{S}^4; K)\), we denote the following group,

\[
E(\mathbb{S}^4; K) = \text{there is an element of degree } 2 \text{ such that } j_K \text{ represents } \text{int}(K) \bmod 2,
\]

and denote a quadratic form (the Rokhlin quadratic form) \( q_K : H_1(K; \mathbb{Z}_2) \to \mathbb{Z}_2 \). Let \( P \) be a compact surface embedded in \( \mathbb{S}^4 \), with its boundary contained in \( K \), normal to \( K \) along its boundary, and its interior is transverse to \( K \). Let \( P^0 \) be a surface transverse to \( P \) obtained by sliding \( P \) parallel to itself over \( K \). Define \( q_K([P]) = \#(\text{int}P \setminus (P^0 \setminus K)) \bmod 2 \), where \( \text{int} \) means the...
interior. This is a well-defined quadratic form with respect to the $\mathbb{Z}_2$-homology intersection form $\langle \cdot, \cdot \rangle_2$ on $K$, i.e. for each pair of elements $x$, $y$ of $H_1(K; \mathbb{Z}_2)$, $q_k(x + y) = q_k(x) + q_k(y) + (x; y)_2$. For the trivial $g$-knot $(S^4; g)$, let $SP_g$ be the subgroup of $M_g$ whose elements leave $q_g$ invariant. This group $SP_g$ is called the spin mapping class group [3]. In the case when $g = 1$, Montesinos showed:

**Theorem 1.1** [10] $E(S^4; 1) = SP_1$.

In this paper, we generalize this result to higher genus:

**Theorem 1.2** For any $g \geq 1$, $E(S^4; g) = SP_g$.

The group $E(S^4; K)$ remains unknown for many non-trivial $g$-knots $K$. On the other hand, for some class of non-trivial $1$-knots $(S^4; K)$, Iwase [6] and the author [5] determined the groups $E(S^4; K)$.

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## 2 Some elements of $E(S^4; g)$

For elements $a$, $b$ and $c$ of a group, we write $c = c^{-1}$, and $a b = a b a$. Here, we introduce a standard form of the trivial $g$-knot $(S^4; g)$. We decompose $S^4 = D_4^+ \cup D_4^-$ and call $S^3 = D_4^+ \setminus D_4^-$ the equator $S^3$, and decompose $S^3 = D_3^+ \cup D_3^-$ and call $S^2 = D_3^+ \setminus D_3^-$ the equator $S^2$. Let $P_g$ be a planar surface constructed from a 2-disk by removing $g$ copies of disjoint 2-disks. As indicated in Figure 1, we denote the boundary components of $P_g$ by $\gamma_0; \gamma_1; \gamma_2; \ldots; \gamma_{2g}$, and denote some properly embedded arcs of $P_g$ by $\gamma_1; \gamma_2; \ldots; \gamma_{2g}$, and denote some properly embedded arcs of $P_g$ by $\gamma_1; \gamma_2; \ldots; \gamma_{2g}$. We parametrize the regular neighborhood of the equator $S^2$ in the equator $S^3$ by $S^2 [-1; 1]$, such that $S^2 f_{0g} = \text{the equator } S^2$, $S^2 [-1; 1] \setminus D_3^+ = S^2 [0; 1]$ and $S^2 [-1; 1] \setminus D_3^- = S^2 [-1; 0]$. We put $P_g$ on the equator $S^2$. Then, $P_g [-1; 1] S^2 [-1; 1]$ is a 3-dimensional handle body, so that, $(S^4; @P_g [-1; 1])$ is the trivial $g$-knot. On $@P_g [-1; 1] = g$.
we denote $c_{2i-1} = \gamma_{2i-1} [-1; 1]$ \((1 \leq i \leq g + 1)\), $b_{2j} = \gamma_{2j} [-1; 1]$ \((1 \leq j \leq g - 1)\), and $c_{2k} = \gamma_{2k}$ \(f \circ g (1 \leq k \leq g)\).

In Figures 2 and 3, these circles are illustrated and some of them are oriented. For a simple closed curve $a$ on $g$, we denote the Dehn twist about $a$ by $T_a$.

The order of composition of maps is the functional one: $T_b T_a$ means we apply $T_b$ then $T_a$.}

First, then $T_b$. We define some elements of $M_g$ as follows:

$$C_i = T_{c_i}; B_i = T_{b_i}; B_0 = T_{b_0};$$

$$X_i = C_{i+1}T_{c_{i+1}}; X_i = C_{i+1}C_{i+1} (1 \ i \ 2g);$$

$$Y_{2j} = C_{2j}B_{2j}C_{2j}; Y_{2j} = C_{2j}B_{2j}C_{2j} (2 \ j \ g - 1);$$

$$D_i = C_i^2 (1 \ i \ 2g + 1);$$

$$D B_{2j} = B_{2j}^2 (2 \ j \ g - 1);$$

$$T = C_1C_3C_5; T_1 = C_1C_3B_4; T_2 = B_4C_5C_7 \ C_{2g+1};$$

When $g \geq 3$, the subgroup of $M_g$ generated by $X_i (1 \ i \ 2g), Y_{2j} (2 \ j \ g - 1), D_i (1 \ i \ 2g + 1), D B_{2j} (2 \ j \ g - 1), T_1,$ and $T_2$ is denoted by $G_g$. It is clear that $X_i$ and $Y_{2j}$ are elements of $G_g$. When $g = 2$, the subgroup of $M_2$ generated by $X_i (1 \ i \ 4), D_j (1 \ j \ 5),$ and $T$ is denoted by $G_2$. For two simple closed curves $l$ and $m$ on $g$, $l$ and $m$ are called $G_g$-equivalent (denote by $l \ G_g m$) if there is an element of $G_g$ such that $(l) = m$. We set

![Figure 4](image)

a basis of $H_1(g; \mathbb{Z})$ as in Figure 4, then for the quadratic form $q_g$ defined in $x_1, q_g(x_i) = q_g(y_i) = 0 (1 \ i \ g)$. By the definitions of $q_g$ and $SP_g$, we have:

**Lemma 2.1** $E(S^4; g) \ SP_g$.

In this section, we show:

**Lemma 2.2** $G_g \ E(S^4; g)$.

As a straightforward corollary of these lemmas, we have:

**Corollary 2.3** $G_g \ SP_g$.

If $G_g \ SP_g$, then Theorem 1.2 is proved. We prove $G_g \ SP_g$ in the next section.
Proof of Lemma 2.2 First we show that, if \( g = 2 \), \( T = C_1 C_3 C_5 \) is an element of \( E(S^4; 2) \). We parametrize the regular neighborhood of the equator \( S^3 \) in \( S^4 \) by \( S^3 \([-1; 1]\) such that \( S^3 f g = \) the equator \( S^3 \) \([−1; 1]\) \( \setminus D_4 = S^3 \([-1; 0]\), and \( S^3 \([-1; 1]\) \( \setminus D_4^+ = S^3 \([0; 1]\). We deform \( g \) in \( S^4 \), in such a way that the surface obtained as a result of this deformation projects onto the equator \( S^3 \) as indicated in Figure 5. In this figure, there are 6 intersecting circles. For each circle, we take two regular neighborhoods \( N_1 \) and \( N_2 \) in \( S^2 \). For \( 0 < t < 1 \), we put \( N_1 \) into \( S^3 \) \( f g \) and \( N_2 \) into \( S^3 \) \( f g \).

This deformation defines an orientation preserving di- eomorphism \( \Psi_1 \) of \( S^4 \).

Let \( r( ) : S^2 ! S^2 \) be the angle rotation whose axis passes through \( N \). We define \( R( ) : S^3 ! S^3 \) by

\[
R( x; t) = (r(t)(x); t) \quad \text{on } S^2 \quad [0; 1]
\]

\[
R( ) = \text{id} \quad \text{on } D^3
\]

\[
R( ) = \text{the angle rotation} \quad \text{on } D^3_+ \quad S^2 \quad [0; 1]
\]
We define an orientation preserving diffeomorphism $\Psi_2$ of $S^4$ by
\[
\Psi_2(x; t) = (R(2 \frac{1-t}{1+t})(x); t) \quad \text{on } S^3 \quad \{\cdot ; \} \\
\Psi_2(x; t) = R(2 \frac{t+1}{1-t})(x); t \quad \text{on } S^3 \quad \{1; \}
\]
\[
\Psi_2(x; t) = R(2 \frac{1-t}{1+t})(x); t \quad \text{on } S^3 \quad \{-1; \}
\]
\[
\Psi_2 = \text{id} \quad \text{on } S^4 - S^3 \quad \{-1; 1\}
\]

Then $\Psi_2^2\Psi_2|_{S^3} = C_1C_3C_5$. In the same way as above, we can show for $g = 3$ that $T_1$ and $T_2$ are elements of $E(S^4; g)$.

Next, for $g = 3$, we show that $X_3 = C_4C_3C_4$ and $D_3 = C_3$ are elements of $E(S^4; g)$. We review a theorem due to Montesinos [10]. We can construct $S^4$ from $B^3 S^1$ and $S^2 D^2$ by attaching their boundary with the natural identification. Let $D^2 S^1$ be the solid torus trivially embedded in $B^3$. We regard $D^2 S^1 S^1 S^1 S^1 S^4$ as the regular neighborhood of a trivial $1$-knot. Let $E^4$ be the exterior of this trivial $1$-knot. The 3 simple closed curves $l = @D^2$, $r = S^1$, $s = S^1$ on $E^4$ represent a basis of $H_1(\mathbb{R}^4; \mathbb{Z})$. Montesinos showed:

**Theorem 2.4** [10, Theorem 5.3] Let $g: E^4 \rightarrow E^4$ be a diffeomorphism which induces an automorphism on $H_1(\mathbb{R}^4; \mathbb{Z})$,
\[
g(l; r; s) = (l; r; s) \otimes \gamma^A
\]
\[
g(1; 0; 1) = (1; 0; 1)
\]

There is a diffeomorphism $G: E^4 \rightarrow E^4$ such that $G_j(\mathbb{R}^4) = g$ if and only if $a = b = 0$ and $a + b + c + d = 0$.

Let $p$ be a point on $S^1 S^1$ disjoint from $r$ and $s$, $N(p)$ be a regular neighborhood of $p$ in the equator $S^3$, then $N = S^1 S^1 - N(p)$ in a regular neighborhood of $r$ and $s$. Figure 6 illustrates deformation of $g$ into $D^2 S^1 S^1$. We bring $s_3$ and $s_4$ to $r$ and $s$ and deform as is indicated by arrows. Then, we can deform $s_3$ in such a way that a regular neighborhood $N^0$ of $s_3$ coincides with $N$ and $s_3 - N^0 \cap N(p)$. Let diffeomorphisms $f_1, f_2$ over $D^2 S^1 S^1$ be defined by $f_1 = \text{id}_{S^2} 1 2 0 1 2 1 0$, $f_2 = \text{id}_{S^2} 2 1 -1 0 0 1 0 1$ (where we present diffeomorphisms on $S^1 S^1$ by its action on the basis $f(r; s)$ of $H_1(S^1 S^1; \mathbb{Z})$ and $r$ and $s$ are oriented as in Figure 6), then $f_1 f_2 = $
Figure 6

\[ C_2^3 = D_3, \quad f_2 | D_2 = C_4 C_3 C_4 \quad = X_3. \]

Since the actions of these homeomorphisms on \( H_1(\mathcal{X}^4; \mathbb{Z}) \) are described by

\[
\begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]

\( (f_1 \mid \mathcal{X}^4)(l;r;s) = (l;r;s) \oplus 0 \quad \text{and} \quad 2A; \)

\[
\begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]

\( (f_2 \mid \mathcal{X}^4)(l;r;s) = (l;r;s) \oplus 0 \quad \text{and} \quad 1A; \)

there are homeomorphisms \( F_1 \) and \( F_2 \) such that \( F_1 \mid D_2 = f_1, \quad F_2 \mid D_2 = f_2 \). These homeomorphisms \( F_1, F_2 \) are extensions of \( f_1, f_2 \) respectively. By the same method as above, we can show that other \( X_i, Y_2, D_1, \) and \( DB_2 \) are elements of \( E(S^4; g) \) for any \( g \geq 2 \).

\[ \square \]

3 A Finite set of generators for the spin mapping class group

In Corollary 2.3, we showed that \( G_g = SP_g \). In this section, we show that \( G_g = SP_g \). That is to say, we show:

**Theorem 3.1** If \( g = 2 \), \( SP_2 \) is generated by \( C_{i+1} C_i C_{i+1} \) (1 i 4), \( C_j^2 \) (1 i 5), and \( C_1 C_3 C_5 \). If \( g = 3 \), \( SP_g \) is generated by \( C_{i+1} C_i C_{i+1} \) (1 i 2g), \( C_2 B_3 C_2 \) (2 j g 1), \( C_k^2 \) (1 k 2g + 1), \( B_i^2 \) (1 l g − 1), \( C_1 C_3 B_4 \) and \( B_4 C_5 C_7 \) C_{2g+1}.

When \( g = 2 \), we use Reidemeister-Schreier’s method to show this. On the other hand, when \( g \geq 3 \), we use other methods. We start from the case when \( g \geq 3 \).
3.1 The hyperelliptic mapping class group

Let \( H_g \) be the subgroup of the mapping class group \( M_g \) generated by \( C_1; C_2; \ldots; C_{2g+1} \). This group is called the hyperelliptic mapping class group. In this group (and also in \( M_g \)), \( C_i \)'s satisfy the following equations:

\[
C_i C_{i+1} C_i = C_{i+1} C_i C_{i+1}; \quad (1 \quad 2g)
\]
\[
C_i C_j = C_j C_i; \quad (j-j \quad 2);
\]

These equations are called braid equation. In this paper, we use these relations frequently. In this section, we show the following lemma for \( H_g \).

**Lemma 3.2** For any \( i = 1; 2; \ldots; 2g+1 \), and any element \( W \) of \( H_g \), \( WC_i C_i W \) is an element of \( G_g \).

**Proof** We call \( C_i \) a positive letter and \( C_i \) a negative letter. A sequence of positive letters is called a positive word. If indices of two letters \( C_i \) and \( C_j \) satisfy \( j_i - j_j = 1 \), then we say \( C_i \) is adjacent to \( C_j \). If there is a negative letter \( B \) in a sequence of letters \( W \), which presents an element of \( H_g \), we replace \( B \) by a sequence of letters \( B B B \). This shows that every element of \( H_g \) is represented by a sequence of positive letters and \( C_j C_j \)'s \((1 \quad j \quad 2g+1)\). If there is a sequence of letters \( XX \) \((X = C_i \ or \ C_i)\) in \( W \), say \( W = W_1 XX W_2 \), then we rewrite

\[
WC_i C_i W = W_1 XX W_2 C_i C_i X X W_1
\]

Therefore, the following claim shows this lemma:

**Claim** For any positive word \( W \) without \( C_j C_j \(1 \quad j \quad 2g+1)\), \( WC_i C_i W \) is an element of \( G_g \).

If the word length of \( W \) is 0, the above claim is trivial. We assume that the word length of \( W \) is at least 1, and we show this claim by the induction on the word length. If the right most letter \( L \) of \( W \) is not adjacent to \( A_i \), and say \( W = W^QL \), then

\[
WC_i C_i W = W^Q C_i C_i C_i W_0 = W^Q C_i L \bigcup C_i W_0 = W^Q C_i C_i W_0:
\]

By the induction hypothesis, \( WC_i C_i W \) is an element of \( G_g \). Therefore, from here to the end of this proof, we assume that the right most letter of \( W \) is adjacent to \( C_i \). Let \( L \) be the word length of \( W \), and \( W = x_1 x_1^{-1} \ldots x_2 x_1 \). The letter \( x_i \) of \( W \) is called a jump, if \( x_{i-1} \) and \( x_i \) are not adjacent. The letter \( x_j \)
of \( W \) is called a turn, if \( x_j \) and \( x_{j-1} \) are not jumps and \( x_j = x_{j-2} \). Considering jumps and turns, we need to show this claim for the following three cases.

**Case 1** When there is not any jump or any turn: Since \( x_i \) and \( x_{i-1} \) are adjacent, \( x_i x_{i-1} x_i^t \) is an element of \( G \). We rewrite

\[
WC_i C_i W = x_i x_{i-1} x_i^t \; x_i x_{i-2} x_{i-3} \; x_1 C_i C_i x_i^t \; x_i^{t-1} x_i^{t-2} x_i \; x_i x_{i-1} x_i^t ;
\]

By the induction hypothesis, \( WC_i C_i W \) is an element of \( G \).

**Case 2** When there are jumps, but there is not any turn: We show in the induction on the number of jumps in \( W \). Let \( x_j \) be the right most jump in \( W \). First we consider the case when \( j = 2 \), say \( W = W^0 x_2 x_1 \). If \( x_2 \) is not adjacent to \( C_i \), we rewrite,

\[
WC_i C_i W = W^0 x_2 x_1 C_i C_i x_i^t x_i^t W^0
\]

By the induction hypothesis on the word length of \( W \), \( WC_i C_i W \) is an element of \( G \). If \( x_2 \) is adjacent to \( C_i \), we rewrite,

\[
WC_i C_i W = W^0 x_2 C_i x_1 x_2 x_i^t W^0
\]

By the induction hypothesis on the word length of \( W \), the first and third terms are elements of \( G \). By the induction hypothesis on the number of jumps in \( W \), the second term is an element of \( G \). Therefore, \( WC_i C_i W \) is an element of \( G \). Next, we consider the case when \( j \) is at least 3. If \( x_j \) is not adjacent to \( x_{j-1} ; \cdots ; x_1 \) then,

\[
W = \cdots : x_j x_{j-1} : \cdots x_1 = \cdots : x_j x_{j-1} : \cdots : x_1,
\]

Therefore, it comes down to the case \( j = 2 \). If there are some letters adjacent to \( x_j \) in \( f x_{j-1} ; \cdots ; x_1 g \), let \( x_i \) be the left most element among them. By the definition of jumps, \( j > i + 1 \), and by the definition of \( x_i \), \( x_j = x_{i-1} \). Therefore,

\[
W = x_j x_i+1 x_i x_{i-1} x_1
\]

\[
= x_i+1 x_j x_i x_{i-1} x_1
\]

\[
= x_i+1 x_{i-1} x_i x_{i-1} x_1
\]

\[
= x_i+1 x_i x_{i-1} x_i x_1
\]
In this subsection, we assume $g = 3$. Therefore, $W = x_1 x_i$ and it comes down to the case $j = 2$.

**Case 3** When there are turns in $W$. Let $x_t$ be the right most turn in $W$. By the definition of turn, $t$ is at least 3. By applying the argument for Case 2 to $x_{t-1} x_{t-2} x_1$, we assume that there is no turn and no jump in $x_{t-1} x_{t-2} x_1$. Since we assume that $x_1$ is adjacent to $C_i$, there may be a case when $x_2 = C_i$.

In that case, we rewrite,

$$WC_i C_i W = x_3 x_2 x_1 C_i x_3 x_2 x_3$$

$$= x_3 x_1 C_i x_3 C_i x_1 x_3$$

$$= x_3 x_1 x_3 C_i x_1 x_3$$

By the induction hypothesis on the word length of $W$, $WC_i C_i W$ is an element of $G_g$. If $x_2 \not\in C_i$, then $x_{t-1} x_{t-2} x_1$ are not adjacent to $C_i$. We rewrite,

$$W = x_1 x_{t-1} x_{t-2} x_{t-3} x_{t-1}$$

$$= x_{t-2} x_{t-1} x_{t-2} x_{t-3} x_{t-1}$$

$$= x_{t-1} x_{t-2} x_{t-1} x_{t-3} x_{t-1}.$$ 

Since we assume that there is no jump and no turn in $x_{t-1} x_{t-2} x_1$, $x_{t-1}$ is not adjacent to $x_{t-3} x_1$. Therefore, $W = x_{t-1} x_{t-2} x_{t-3} x_{t-1} x_{t-1}$.

With remarking that $x_{t-1}$ is not adjacent to $C_i$, we rewrite,

$$WC_i C_i W = x_{t-1} x_{t-2} x_{t-3} x_{t-1} C_i x_{t-1} C_i x_{t-1} x_{t-3} x_{t-2} x_{t-1}$$

$$= x_{t-1} x_{t-2} x_{t-3} x_{t-2} x_{t-1} C_i x_{t-1} x_{t-3} x_{t-2} x_{t-1}$$

$$= x_{t-1} x_{t-2} x_{t-3} x_{t-1} C_i x_{t-1} x_{t-3} x_{t-2} x_{t-1}.$$ 

By the induction hypothesis on the word length of $W$, $WC_i C_i W$ is an element of $G_g$.

**3.2 The Torelli group $l_g$**

In this subsection, we assume $g = 3$. There is a natural surjection $\phi : M_g \rightarrow \text{Sp}(2g; \mathbb{Z})$ defined by the action of $M_g$ on the group $H_3(\mathbb{Z})$. We denote the kernel of $\phi$ by $l_g$ and call this the Torelli group. In this subsection, we prove the following lemma:

**Lemma 3.3** The Torelli group $l_g$ is a subgroup of $G_g$. 

Johnson [7] showed that, when \( g \) is larger than or equal to 3, \( I_g \) is finitely generated. We review his result. We orient and call simple closed curves as indicated in Figure 2, and call \((c_1; c_2; \ldots ; c_{2g+1})\) and \((c_5; c_6; \ldots ; c_{2g+1})\) as chains. For oriented simple closed curves \( d \) and \( e \) which mutually intersect in one point, we construct an oriented simple closed curve \( d + e \) from \( d \) and \( e \) as follows: choose a disk neighborhood of the intersection point and in it make a replacement as indicated in Figure 7. For a consecutive subset \( (c_1; c_2; \ldots ; c_{2g+1}) \) of a chain, let \( c_1 + \ldots + c_i \) be the oriented simple closed curve constructed by repeated applications of the above operations. Let \((i_1; \ldots ; i_{r+1})\) be a subsequence of \((1; 2; \ldots ; 2g + 1)\) (resp. \((5; \ldots ; 2g + 1)\)). We construct the union of circles \( C = c_1 + \ldots + c_{2g-1} \) if \( r \) is odd, \( j \) and \( j + 1 \) are either both contained in or are disjoint from the \( i \)'s. Let \( \gamma \) be the element of \( M_g \) defined as the composition of the positive Dehn twist along the boundary curve to the left of \( C \) and the negative Dehn twist along the boundary curve to the right of \( C \). Then, \( \gamma \) is an element of \( I_g \). We denote \( \gamma \) by \([i_1; \ldots ; i_{r+1}]\), and call this the odd subchain map of \((c_1; c_2; \ldots ; c_{2g+1})\) (resp. \((c_5; c_6; \ldots ; c_{2g+1})\)). Johnson [7] showed the following theorem:

**Theorem 3.4** [7, Main Theorem] For \( g \geq 3 \), the odd subchain maps of the two chains \((c_1; c_2; \ldots ; c_{2g+1})\) and \((c_5; c_6; \ldots ; c_{2g+1})\) generate \( I_g \).

We use the following results by Johnson [7].

**Lemma 3.5** [7] (a) \( C_j \) commutes with \([i_1; i_2; \ldots ; i_{r+1}]\) if and only if \( j \) and \( j + 1 \) are either both contained in or are disjoint from the \( i \)'s.
(b) If \( i \notin j + 1 \), then \( C_j \) \([i_1; \ldots ; i_{r+1}] = [i_1; \ldots ; i_{r+1}] \) and \( C_j \) \([i_1; \ldots ; i_{r+1}] = [i_1; \ldots ; i_{r+1}] \).
(c) If \( k \notin j \), then \( C_j \) \([i_1; \ldots ; i_{r+1}] = [i_1; \ldots ; i_{r+1}] \) and \( C_j \) \([i_1; \ldots ; i_{r+1}] = [i_1; \ldots ; i_{r+1}] \).
(d) \([1; 2; 3; 4] [1; 2; 5; 6; \ldots ; 2n] B_4 [3; 4; 5; \ldots ; 2n] = [5; 6; \ldots ; 2n] \) where \( 3 \leq n \leq 9 \).

First we show that some odd subchain maps are elements of \( G_g \).
Lemma 3.6  [1; 2; 3; 4], [1; 3; 5; 7; \ldots ; 2i + 1; \ldots ; 2n - 1] (n is even, and 4 \ n \ g + 1), and [1; 2; 4; 6; \ldots ; 2i; \ldots ; 2n - 2] (n is even, and 4 \ n \ g + 2) are elements of $G_g$.

Proof  In this proof, for a sequence $f_i g$ of elements of $M_g$, we write,

\[
\gamma^n_{f_i} = \begin{pmatrix} f_n f_{n+1} & f_m; & n \ m; \\
1 \ n \end{pmatrix}
\]

(1) [1; 2; 3; 4] is an element of $G_g$: [1; 2; 3; 4] is equal to $B_4 B_4$. Since $C_4 C_3 C_2 C_1 C_2 C_3 C_4 (b_k) = b_k$,

\[
[1; 2; 3; 4] = B_4 C_4 C_3 C_2 C_1 C_2 C_3 C_4 B_4 C_4 C_3 C_2 C_1 C_2 C_3 C_4
\]

Therefore, [1; 2; 3; 4] is an element of $G_g$.

(2) [1; 3; 5; 7; \ldots ; 2i + 1; \ldots ; 2n - 1] (n is even, and 4 \ n \ g + 1) are elements of $G_g$: By (b) of Lemma 3.5,

\[
[1; 3; 5; 7; \ldots ; 2i + 1; \ldots ; 2n - 1] = ( \gamma^n_{f_i} \gamma^k \gamma^{k-1} \gamma^{k-2} \cdots \gamma^1 \gamma^0 ) [1; 2; 3; 4; \ldots ; n]
\]

Since $[1; 2; 3; 4; \ldots ; n] = B_n B_2 B_2$, and $b_i = Q_n \gamma^i C_i = c_1 C_1 Q_n \gamma^i C_i (b_i)$,

\[
[1; 2; 3; 4; \ldots ; n] = B_n C_i C_1 C_1 C_1 B_n \gamma^n_{i=n} C_i C_1 C_1 \gamma^n_{i=2} C_i C_1 C_1 \gamma^n_{i=1} C_i C_1 C_1 \gamma^n_{i=0} C_i C_1 C_1 \gamma^n_{i=0} C_i C_1 C_1
\]

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Therefore,

\[ [1; 3; 5; 7; \ldots ; 2n - 1] = \gamma^n \gamma^1 \gamma^n \gamma^{i-1} \left( \gamma^k \left( \gamma^l C_i B_n C_i \right) (C_{k-1}C_{k-1}) \right) g \]

\[ k=2 \quad l=n-1 \quad i=2l \quad i=n \]

\[ \gamma^1 \gamma^n \gamma^{i-1} \left( \gamma^k \left( \gamma^l C_i B_n \right) (C_nC_n) \right) \]

\[ l=n-1 \quad i=2l \quad i=n \]

\[ \gamma^n \gamma^1 \gamma^n \gamma^{i-1} \left( \gamma^k \left( \gamma^l C_i C_j \right) (C_{k-1}C_{k-1}) \right) g \]

\[ k=2 \quad l=n-1 \quad i=2l \quad i=n \]

\[ \gamma^1 \gamma^n \gamma^{i-1} \left( \gamma^k \left( \gamma^l C_i \right) (\gamma^l C_i \gamma^l C_n) \right) ; \]

By Lemma 3.2, \( Q^n \sum_{k=1}^{n} (Q^1 \sum_{i=2}^{n} (Q^1 C_i C_i) (C_{k-1}C_{k-1}) \right) g \)

\( Q^n \sum_{i=2}^{n} (Q^1 C_i C_i) (C_{k-1}C_{k-1}) \) are elements of \( G_g \). By braid relations for \( M_g \), (in the following equations \( j = n - 1 \))

\[ \gamma^n \gamma^{i-1} \left( \gamma^k \left( \gamma^l C_i C_j \right) (C_{k-1}C_{k-1}) \right) \]

\[ (C_{j-1}C_i) \left( C_{j-1}C_{j-1} \right) = C_{j-1}C_i \gamma^n \gamma^{i-1} \]

\[ C_i \left( C_{j-1}C_{j-1} \right) C_i \gamma^n \gamma^{i-1} \]

\[ C_i \left( C_{j-1}C_{j-1} \right) C_i \gamma^n \gamma^{i-1} \]

\[ C_i \left( C_{j-1}C_{j-1} \right) C_i \gamma^n \gamma^{i-1} \]

\[ (C_{n-1}C_n) (C_{n-1}C_{n-1}) = C_{n-1}C_n C_{n-1} \gamma^n \gamma^{i-1} \]

\[ = C_{n-1} \gamma^n \gamma^{i-1} \]

By the above equation and the fact that \( B_n \) commutes with \( C_j \) (1 \( j \) \( n - 1 \)),

\[ (B_n \gamma^n \gamma^{i-1} \gamma^2) \left( C_{k-1}C_{k-1} \right) = \left( C_j \gamma^n \gamma^{i-1} \gamma^2 \right) \left( C_{1}C_{1} \right) \gamma^2 \]

\[ \gamma^n \gamma^{i-1} \gamma^2 \left( C_{1}C_{1} \right) \gamma^2 \]

\[ \gamma^n \gamma^{i-1} \gamma^2 \left( C_{1}C_{1} \right) \gamma^2 \]

Since, for \( j = k-2 \), \( \gamma^n \gamma^{i-1} \gamma^2 \left( C_{1}C_{1} \right) \gamma^2 \]

\[ \gamma^n \gamma^{i-1} \gamma^2 \left( C_{1}C_{1} \right) \gamma^2 ; \]

we obtain,

\[
\left( \begin{array}{cccc}
Y_l & nY_l^{i-1} & Y_l & Y_l \\
C_i & C_i & B_n & C_i \\
\end{array} \right)
\]

Therefore, for showing that \([1; 3; 5; \cdots; 2n-1]\) is an element of \(G_g\), it suffices to show that \((Q_1^{1}Q_{n+1}^{1}C_i)B_n^{2}Q_{i=1}^{n+1}C_i\) is an element of \(G_g\). Figure 8 illustrates \(u = (X_5X_3X_1)\). We investigate the action of elements of \(G_g\) on \(u\). As indicated in Figure 9, \(X_5X_3X_1\) acts on \(u\). We make \(Q_2^{2}X_{i=2}^{n-2}X_{i+1}^{4i+1}X_{4i-1}\) act on this circle. In the middle of this action, \(X_{i=2}^{4i+1}X_{4i-1}\) acts locally as in Figure 10. Hence, \(Q_2^{2}X_{i=2}^{n-2}X_{i=2}^{n+1}X_{i=1}^{n-1}\) acts on \(u\). We make \(Q_2^{2}X_{i=1}^{n+1}X_{i}^{4i-1}\) act on this circle. In the middle of this action, \(X_{i=1}^{4i}X_{i+1}^{4i-1}\) acts locally as in Figure 11. This figure shows that, by the action of \(X_{i=1}^{4}X_{i=2}^{4}X_{i=3}^{2n-3}X_{i=4}^{2n-5}\), this curve is changed to the \(u\) of \(n-4\). Therefore, for our purpose, it suffices to show that \(T_uT_u\) is an element of \(G_g\) only for \(n = 4\) or \(n = 6\). Figure 12 shows that, when \(n = 4\), \(T_uT_u = (X_1^{1}X_2^{4}X_3^{5})(Y_4^{4}Y_4^{4})\).
Figure 13 shows that, when \( n = 5 \),

\[ T_u T_u = (X_1 X_3 X_5 X_7 X_9 Y_6 X_4 X_6)\quad D_8. \]
(3) $[1; 2; 4; 6; \ldots; 2l; \ldots; 2n - 2]$ (n is even, and $4 \leq g + 2$) are elements of $G_g$. By (b) of Lemma 3.5,

$$[1; 2; 4; 6; 8; \ldots; 2n - 2] = (\frac{Y^l}{n^{Y-1}} \frac{C_i}{n^2 \mathcal{C}_i}) [1; 2; 3; 4; \ldots; n]:$$

In the same way as (2),

$$[1; 2; 4; 6; 8; \ldots; 2n - 2] = f(\frac{Y^l}{n^{Y-1}} \frac{\mathcal{C}_i}{B_n} \frac{C_i}{C_l}) (C_{k-1}C_{k-1})g$$

By Lemma 3.2, $(\frac{Q}{l=n-2} \frac{Q_{n+l-1}}{i=2l+1 \mathcal{C}_i} \frac{Q_k}{i=n \mathcal{C}_i}) (C_{k-1}C_{k-1})$ and $(\frac{Q}{l=n+2} \frac{Q_{n+l-1}}{i=2l+1 \mathcal{C}_i} \frac{C_l}{C_n}) (C_{k-1}C_{k-1})$ are elements of $G_g$. By the same method as in (2), but using

$$\frac{Y^l}{n^{Y-1}} \frac{C_j}{C_i} = (\frac{C_{2j-1}C_{2j-2}C_{2j-1}}{C_{C_{k-1}}} C_1) \frac{Y^l}{n^{Y-1}} \frac{C_i}{C_j};$$

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\begin{align*}
\gamma_l \quad n \gamma_l & = \gamma_l \\
\gamma_j \quad C_i & = \gamma_l \quad (C_{2j} - 1 C_{2j-1}) \\
\gamma_l \quad n \gamma_l & = \gamma_l \quad C_i \\
\end{align*}

we conclude that, for our purpose, it suffices to show that \( C_i \) and \( (C_i, C_i) \) are elements of \( G_g \). Figure 14 illustrates \( v = \circ \cdots \circ \) and \( w = \circ \cdots \circ \).

Figure 14

and \( w = C_1(v) \). First we investigate the actions of elements of \( G_g \) on \( v \). In the following argument, we will refer the pictures in Figure 15 and Figure 18 by the number with ( ). By the action of \( T_2 \), \( v \) is changed to \( 0 \). Now, we show (1) is \( G_g \)-equivalent to (6). (1) is altered to (2) by the action of \( Y_6 \).

We make a sequence of \( X_{4i+1} X_{4i-1} \)'s act on this circle. In the middle of this process, each \( X_{4i+1} \) acts locally as indicated in Figure 16. Hence, (2) is \( G_g \)-equivalent to (3). By the action of \( X_{4m-1} \), (3) is deformed to (4). In the middle of a sequential action of \( X_{4i+1} X_{4i-1} \)'s, each \( X_{4i+3} X_{4i-1} \) acts locally as shown in Figure 17. Hence, (4) and (5) are \( G_g \)-equivalent. As a result of the action of \( X_{4m-3} \), (5) is altered to (6). The above argument shows that (1) is \( G_g \)-equivalent to (6). For (0), we apply the above process from (1) to (6) repeatedly, then we get (7). The element \( X_5 X_7 Y_6 \) alters (7) into (8). If \( \frac{n}{2} \) is even, \( DB_{4}^{\frac{n}{2}-1} \) deforms (8) into (9). Since (9) is changed to (10) by the action of \( X_3 \), there exists an element \( h \) of \( G_g \) such that \( h (T_v, T_v) = X_1 X_1 \). If \( \frac{n}{2} \) is odd, \( DB_{4}^{\frac{n}{2}+2} \) deforms (8) into (11). Since (11) is changed to (12) by the action

Figure 15

(0)  
\[ k \text{ twists} \]

(1)  
\[ 2m \]
\[ k \text{ twists} \]

(2)  
\[ k-1 \text{ twists} \]

(3)  
\[ k-1 \text{ twists} \]

(4)  
\[ k-1 \text{ twists} \]

(5)  
\[ k-1 \text{ twists} \]

(6)  
\[ 2m-2 \]

Figure 15

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Figure 16

Figure 17

Figure 18

of $X_1Y_4$, there exists an element $h$ of $G_g$ such that $h(T_vT_v) = D_3$. Next, we investigate the actions of $G_g$ on $w$. The action of $T_1T_2$ deforms $w$ into (1) of Figure 19. After the repeated application of the actions from (1) to (6) of Figure 15, this circle is altered to (2) of Figure 19. By the same argument for $v$, when $\frac{n}{2}$ is even, there is a $h$ of $G_g$ such that $h(T_wT_w) = D_3$, on the other hand, when $\frac{n}{2}$ is odd, there is a $h$ of $G_g$ such that $h(T_wT_w) = X_1X_1$. Therefore, $[1;2;4;6;8;\cdots;2n−2]$ is an element of $G_g$.

We prove that any odd subchain map of $(c_1;c_2;c_3;\cdots;c_{2g+1})$ or $(c;c_5;c_6;\cdots;c_{2g})$ is a product of elements listed on Lemma 3.6 and elements of $G_g$. The following lemma shows that any odd subchain map of $(c;c_5;c_6;\cdots;c_{2g})$ is a product of an odd subchain map of $(c_1;c_2;c_3;\cdots;c_{2g+1})$ and elements of $G_g$. 

Lemma 3.7  \( D_3 T_1 \) \( (c) = c_3 + c_4 \).

**Proof**  Figure 20 proves this lemma.

From here to the end of this subsection, odd subchain maps mean only those of \((c_1; c_2; c_3; \cdots; c_{2g+1})\). The following lemma shows that any odd subchain map, whose length is at least 5 and which begins from \(1; 2; 3; 4\), is a product of shorter odd subchain maps and elements of \(G_g\).

**Lemma 3.8**

\[
[1; 2; 3; 4][1; 2; 3; 5]^{-1}[1; 2; 3; 4][1; 2; 4; 6; 7; \cdots; 2n]
\]

\[
(C_4 B_4 \overline{C_4}) \quad [3; 4; 5; \cdots; 2n] = [4; 6; 7; \cdots; 2n][1; 2; 3; 4; \cdots; 2n]
\]

**Proof**  By (a) of Lemma 3.5, \( C_4 \quad [3; 4; 5; \cdots; 2n] = [3; 4; 5; \cdots; 2n] \), and by (d) of Lemma 3.5,

\[
[1; 2; 3; 4][1; 2; 5; 6; \cdots; 2n] \quad (B_4 \overline{C_4}) \quad [3; 4; 5; \cdots; 2n] = [5; 6; \cdots; 2n][1; 2; 3; 4; \cdots; 2n]
\]

By applying \(C_4\) to the above equation, we get the equation which we need.
For any odd subchain map \([i_1; i_2; \ldots; i_r]\), we denote a sequence \([1; 2; \ldots; 2g+2]\) as follows: \(k = 1\) if \(k\) is a member of \(i_1; i_2; \ldots; i_g\), and \(k = 0\) if \(k\) is not a member of \(i_1; i_2; \ldots; i_g\). For this sequence \([1; 2; \ldots; 2g+2]\), we construct the sequence \([1; 2; \ldots; 2g+2]\) by the following rule: \((2i - 1; 2i) = (0; 0)\) if \((2i - 1; 2i) = (0; 0)\), \((2i - 1; 2i) = (1; 0)\) if \((2i - 1; 2i) = (0; 1)\), \((2i - 1; 2i) = (0; 1)\) if \((2i - 1; 2i) = (1; 0)\), \((2i - 1; 2i) = (1; 1)\) if \((2i - 1; 2i) = (1; 1)\). The odd subchain map \([i_1; i_2; \ldots; i_r]\), which corresponds to the sequence \([1; 2; \ldots; 2g+2]\), is called the reversion of \([i_1; i_2; \ldots; i_r]\).

**Lemma 3.9**  
(1) For any odd subchain map \(c\), there is an element of \(G_g\) which brings \(c\) to its reversion.
(2) When \(k = i - 3\), \((C_{i-1}C_{i-2}C_{i-1}) [\ldots; k; i; j; \ldots] = [\ldots; k; i - 2; j; \ldots]\).
(3) When \(k = i - 2\), \((C_iC_{i-1}C_i) [\ldots; k; i; 1; \ldots] = [\ldots; k; i - 1; i; \ldots]\).

**Proof**  
Lemma 3.5 shows (2) and (3). Since, \(T_1T_2 = C_2C_3C_5C_{2g+1}\) and \(D_{2i-1} = C_{2i-1}C_{2i-1}(1 + g + 1)\) are elements of \(G_g\), \(C_1C_3C_5C_{2g+1}\) is an element of \(G_g\) for any choice of \(1\)'s. Let \([1; 2; \ldots; 2g+2]\) be the 0-1 sequence corresponding to \([i_1; i_2; \ldots; i_r]\). We denote \(\gamma_i (1 + g + 1)\) as follows: \(\gamma_i = +1\) if \((2i - 1; 2i) = (0; 0); (0; 1)\), or \((1; 1)\), and \(\gamma_i = -1\) if \((2i - 1; 2i) = (1; 0)\). Then \((C_1C_3C_5C_{2g+1}) [i_1; \ldots; i_r]\) is the reversion of \([i_1; \ldots; i_r]\).  

By (2) of the above lemma, any odd subchain map is deformed to an odd subchain map \([i_1; i_2; \ldots; i_r]\) such that \(i_{l+1} - i_l = 2\) under the action of \(G_g\). If there are at least two disjoint pairs of indices \(l_l; i_{l+1}\) in an odd subchain map \([i_1; i_2; \ldots; i_r]\) such that \(i_{l+1} = i_l + 1\), then, by (3) of the above lemma, this odd subchain map is altered to the odd subchain map which begins from \(1; 2; 3; 4\) under the action of \(G_g\). Therefore, by Lemma 3.8, this odd subchain map is a product of shorter odd subchain maps and elements of \(G_g\). Hence, it suits to show that \([1; 3; 5; 7; 9; \ldots]\), \([2; 4; 6; 8; \ldots]\), \([1; 2; 3; 5; 7; \ldots]\), \([1; 2; 4; 6; 8; \ldots]\), \([1; 2; 3; 4]\) are elements of \(G_g\). By (1) of Lemma 3.9, the second ones are changed to the odd ones, and the third ones are changed to the fourth ones by the action of \(G_g\). On the other hand, we have already shown that \([1; 3; 5; 7; 9; \ldots]\), \([1; 2; 4; 6; 8; \ldots]\), and \([1; 2; 3; 4]\) are elements of \(G_g\) in Lemma 3.6. Therefore, Lemma 3.3 is proved.

### 3.3 The level 2 prime congruence subgroup of \(\text{Sp}(2g; \mathbb{Z})\)

In this subsection, we assume \(g = 3\). Let \(\phi\) be the natural homomorphism from \(M_g\) to \(\text{Sp}(2g; \mathbb{Z}_2)\) defined by the action of \(M_g\) on the \(\mathbb{Z}_2\)-coefficient homology group \(H_1(\text{Sp}_g; \mathbb{Z}_2)\). In this section, we show the following lemma.
Lemma 3.10 \( \ker \ Z_2 \) is a subgroup of \( G_g \).

We denote the kernel of the natural homomorphism from \( \text{Sp}(2g; \mathbb{Z}) \) to \( \text{Sp}(2g; \mathbb{Z}_2) \) by \( \text{Sp}^{(2)}(2g) \). We set a basis of \( H_1( g; \mathbb{Z}) \) as in Figure 4, and define the intersection form \( (, ) \) on \( H_1( g; \mathbb{Z}) \) to satisfy \( (x_i; y_j) = i \cdot (x_i; x_j) = (y_i; y_j) = 0 \) \((1 \leq i, j \leq g)\). An element \( a \) of \( H_1( g; \mathbb{Z}) \) is called primitive if there is no element \( n(\neq 0, 1) \) of \( \mathbb{Z} \), and no element \( b \) of \( H_1( g; \mathbb{Z}) \) such that \( a = nb \). For a primitive element \( a \) of \( H_1( g; \mathbb{Z}) \), we define an isomorphism \( T_a : H_1( g; \mathbb{Z}) \to H_1( g; \mathbb{Z}) \) ! \( H_1( g; \mathbb{Z}) \) by \( T_a(v) = v + (a; v)a \). This isomorphism is the same as the action of \( \text{Dehn twist about a simple closed curve representing a} \). Johnson [8] showed the following result.

Lemma 3.11 \( \text{Sp}^{(2)}(2g) \) is generated by square transvections.

Lemma 3.12 \( \text{Sp}^{(2)}(2g) \) is generated by the square transvections about the primitive elements \( g \sum_{i=1}^{g} (x_i + iy_i) \), where \( i = 0, 1 \) and \( i = 0, 1 \).

We define, for any primitive element \( a \) and \( b \) of \( H_1( g; \mathbb{Z}) \), two operation \( \Box \) and \( \boxplus \) by

\[
\begin{align*}
\Box b &= a + 2(a; b)b, \\
\boxplus b &= a - 2(a; b)b.
\end{align*}
\]

We remark that \( T_{a \Box b}^2 = T_{b}^{-2}T_{a}^2T_{b}^2 \), \( T_{a \boxplus b}^2 = T_{b}^{-2}T_{a}^2T_{b}^{-2} \), and \( (a \boxplus b) \Box b = a = (a \Box b) \boxplus b \). We denote the element \( g \sum_{i=1}^{g} (x_i + iy_i) \) of \( H_1( g; \mathbb{Z}) \) by \( [(a_1^1; a_1^2); (a_2^1; a_2^2); \ldots; (a_g^1; a_g^2)] \), and call each \( (a_i^1; a_i^2) \) as a block. For a positive integer \( k \), \( a \boxplus b \) is the result of the \( k \)-fold application of \( \Box b \) on \( a \), and \( a \boxplus b \) is the result of the \( k \)-fold application of \( \Box b \) on \( a \).

Lemma 3.13 For any primitive element \( a \) of \( H_1( g; \mathbb{Z}) \), by applying \( \Box \) \( n \) times \((0; 0); \ldots; (0; 0); (1; 0); (0; 0); \ldots; (0; 0) \) or \( \Box \) \( n \) times \((0; 0); \ldots; (0; 0); (0; 0); (1; 0); (0; 0); \ldots; (0; 0) \) several times, each block of \( a \) is altered to \((0; 0), (p; 0), (0; p), \) or \((p; p) \).

Proof Let \((m; n)\) be the \( i \)-th block of \( a \). First we consider the case when \( jmn > jn \notin 0 \). There is an integer \( k \) such that \( jmn - 2knj > jn \). Let \( e_i \) be the element of \( H_1( g; \mathbb{Z}) \), the \( i \)-th block of which is \((1; 0) \), and every other block of which is \((0; 0) \). Since, \( \Box [ ; (m; n); ] \boxplus e_i = [ ; (m - 2n; n); ] \), and \( \Box [ ; (m; n); ] \boxplus e_i = [ ; (m + 2n; n); ] \\text{we get} \ [ ; (m; n); ] \boxplus e_i k = [ ; (m - 2kn; n); ] \). This means that, by repeated application of \( \Box e_i \), the \( i \)-th block \((m; n) \) is altered such that \( jmn > jn \). Next, we consider the case when...
0 \leq jm < jn. Let $f_i$ be the element of $H_1(\mathbb{Z}, \mathbb{Z})$, the $i$-th block of which is $(0, 1)$, and other blocks of which are $(0, 0)$. Since, \[ \begin{bmatrix} (m; n) \end{bmatrix} f_i = \begin{bmatrix} (m; n + 2m) \end{bmatrix} \] and \[ \begin{bmatrix} (m; n) \end{bmatrix} f_i = \begin{bmatrix} (m; n - 2m) \end{bmatrix} \] by the same argument as the previous case, by repeated application of $f_i$, the $i$-th block is altered such that $jm = jn$. The above arguments show that, after several application of $e_q$ or $f_i$, the $i$-th block $(m; n)$ of $a$ is altered to be $jm = jn$, or $m = 0$, or $n = 0$. If $n = -m$, the $i$-th block changed to $(m; m)$ by the application of $f_i$. For each $i$-th block, we do the same operation as above. Then, this lemma follows.

For a primitive element of $H_1(\mathbb{Z}, \mathbb{Z})$, each of whose blocks is $(p; 0)$, or $(0; p)$, or $(p; p)$, (where $p$ can be different from block to block) we apply several operations $\begin{bmatrix} 1, \ldots, i, j, k, \ldots \end{bmatrix}$, where $i = 0, 1$ and $j = 0, 1$. Then we obtain the following equations, where $\square$ means a sequence of $(0; 0)$, and $\sqcap$ means the part which is not changed.

\[
\begin{align*}
\square & \begin{bmatrix} (1; 0); (0; 1) \end{bmatrix} \sqcap \begin{bmatrix} (0; 0); (0; 1) \end{bmatrix} \\
= & \begin{bmatrix} (1; 0); (q; 0) \end{bmatrix} \\
\square & \begin{bmatrix} (1; 0); (0; 1) \end{bmatrix} \sqcap \begin{bmatrix} (0; 0); (0; 1) \end{bmatrix} \\
= & \begin{bmatrix} (1; 0); (q; 0) \end{bmatrix} \\
\square & \begin{bmatrix} (1; 0); (0; 1) \end{bmatrix} \sqcap \begin{bmatrix} (0; 1); (0; 1) \end{bmatrix} \\
= & \begin{bmatrix} (1; 0); (q; 0) \end{bmatrix} \\
\square & \begin{bmatrix} (1; 0); (0; 1) \end{bmatrix} \sqcap \begin{bmatrix} (0; 1); (0; 0) \end{bmatrix} \\
= & \begin{bmatrix} (1; 0); (q; 0) \end{bmatrix} \\
\square & \begin{bmatrix} (1; 0); (0; 1) \end{bmatrix} \sqcap \begin{bmatrix} (0; 1); (0; 0) \end{bmatrix} \\
= & \begin{bmatrix} (1; 0); (q; 0) \end{bmatrix} \\
\square & \begin{bmatrix} (1; 0); (0; 1) \end{bmatrix} \sqcap \begin{bmatrix} (0; 1); (0; 0) \end{bmatrix} \\
= & \begin{bmatrix} (1; 0); (q; 0) \end{bmatrix} \\
\square & \begin{bmatrix} (1; 0); (0; 1) \end{bmatrix} \sqcap \begin{bmatrix} (0; 1); (0; 0) \end{bmatrix} \\
= & \begin{bmatrix} (1; 0); (q; 0) \end{bmatrix} \\
\end{align*}
\]

Lemma 3.14 For any primitive element $a$ of $H_1(g;\mathbb{Z})$, by applying $\Box[(1; 1); (q; q)]$ (where $i = 0; 1$, and $i = 0; 1$) several times, $a$ is
deformed to $[1; 1]; (g; g)]$ (where $i = 0; 1$, and $i = 0; 1$) or $[(-1; 0); 1]$. □

Since $T^2(v) = v + 2(-a; v)(-v) = v + 2(a; v)v = T^2_0(v)$, we do not need to consider the elements $[-1; 0]$. Hence, Lemma 3.12 follows.

For each element $[(1; 1); (g; g)]$ (where $i = 0; 1$, and $i = 0; 1$) of $H_1(g; \mathbb{Z})$, we construct an oriented simple closed curve on $g$ which represent this homology class. For each $i$-th block, if $(i; i) = (0; 0)$, we prepare (0) of Figure 21, if $(i; i) = (0; 1)$, we prepare (1) of Figure 21, if $(i; i) = (1; 1)$, we prepare (2) of Figure 21, if $(i; i) = (1; 0)$, we prepare (3) of Figure 21. After that, we glue them along the boundaries and cap the left boundary component by (-) of Figure 21 and the right boundary component by (+) of Figure 21. We denote this oriented simple closed curve on $g$ by $f((1; 1); (g; g))$. Here, we remark that the action of $T_f((1; 1); (g; g))$ on $H_1(g; \mathbb{Z})$ equals $T_f((1; 1); (g; g))$, and, for any $f$ of $M_g$, $T_f((1; 1); (g; g)) = T_f((1; 1); (g; g))$.

**Lemma 3.15** For any $f((1; 1); (g; g))$, there is an element of $G_g$ such that

$$(f((1; 1); (g; g)) = f((0; 1); (0; 0)); (0; 0); ; (0; 0))$$

or $f((1; 1); (0; 0)); (0; 0); ; (0; 0))$

or $f((0; 0); (1; 1); (0; 0); ; (0; 0))$.

**Proof** If the $i$-th block is (3), by the action of $\mathbb{Z}_2$, this block is changed to (1). Therefore, it suffices to show this lemma in the case when each block is not (3). First we investigate actions of elements of $G_g$ on adjacent blocks, say the $i$-th block and the $i + 1$-st block. Each picture of Figure 22 shows the action...
On homeomorphisms over surfaces trivially embedded in the 4-sphere

Figure 22

of $G_g$ on this adjacent blocks.

(a) shows $f$ ; $(0; 0); (0; 1); g f ; (0; 1); (0; 0); g$
(b) shows $f$ ; $(0; 0); (1; 1); g f ; (1; 1); (0; 1); g$
(c) shows $f$ ; $(1; 1); (1; 1); g f ; (0; 1); (0; 0); g$
(d) shows $f$ ; $(0; 1); (0; 1); g f ; (0; 1); (0; 0); g$
(e) shows $f$ ; $(0; 1); (1; 1); g f ; (1; 1); (0; 0); g$

For an oriented simple closed curve $x = f (1; 1); (0; 0); g (g, g) g$, each of whose block is $(0; 0)$ or $(0; 1)$ or $(1; 1)$, let the right most non-$(0; 0)$ block be the $j$-th block. By the induction on $j$, we show that $x$ is $G_g$-equivalent to $f (0; 1); (0; 0); (0; 0); (0; 0) g$ or $f (1; 1); (0; 0); (0; 0); (0; 0) g$ or $f (0; 0); (1; 1); (0; 0); (0; 0)$.
(0; 0)g. If j = 1, it is trivial.
When the j-th block is (0; 1). If each block between the \( \text{rst} \) block and the \((j - 1)\)-st block is (0; 0), then, by repeated application of (a), \( x \) is \( G_3 \)-equivalent to \( f(0; 1); (0; 0); \cdots; (0; 0)g \). If there is a block between the \( \text{rst} \) block and the \((j - 1)\)-st block which is not (0; 0), by the induction hypothesis, the sequence from the \( \text{rst} \) block to the \((j - 1)\)-st block is \( G_3 \)-equivalent to \( (0; 1); (0; 0); (0; 0); \cdots; (0; 0) \) or \((1; 1); (0; 0); (0; 0); \cdots; (0; 0)\) or \((0; 0); (1; 1); (0; 0); \cdots; (0; 0)\). In the \( \text{rst} \) case,

\[
x_{G_3} f(0; 1); (0; 0); (0; 0); \cdots; (0; 0); (0; 1); (0; 0)g (\text{by the hypothesis})
\]

\[
g_{G_3} f(0; 1); (0; 1); (0; 0); \cdots; (0; 0); (0; 0)g (\text{by (a)})
\]

\[
g_{G_3} f(0; 1); (0; 0); (0; 0); \cdots; (0; 0)g (\text{by (d)})
\]

In the second case,

\[
x_{G_3} f(1; 1); (0; 0); (0; 0); \cdots; (0; 0); (0; 1); (0; 0)g (\text{by the hypothesis})
\]

\[
g_{G_3} f(1; 1); (0; 1); (0; 0); \cdots; (0; 0); (0; 0)g (\text{by (a)})
\]

\[
g_{G_3} f(0; 0); (1; 1); (0; 0); \cdots; (0; 0)g (\text{by (b)})
\]

In the third case,

\[
x_{G_3} f(0; 0); (1; 1); (0; 0); \cdots; (0; 0); (0; 1); (0; 0)g (\text{by the hypothesis})
\]

\[
g_{G_3} f(0; 0); (1; 1); (0; 1); \cdots; (0; 0); (0; 0)g (\text{by (a)})
\]

\[
g_{G_3} f(1; 1); (0; 1); (0; 1); \cdots; (0; 0); (0; 0)g (\text{by (b)})
\]

\[
g_{G_3} f(1; 1); (0; 1); (0; 0); \cdots; (0; 0)g (\text{by (d)})
\]

\[
g_{G_3} f(0; 0); (1; 1); (0; 0); \cdots; (0; 0)g (\text{by (b)})
\]

When the j-th block is (1; 1). If every block between the \( \text{rst} \) block and \((j - 1)\)-st block is (0; 0), then,

\[
x_{G_3} f(1; 1); (0; 1); (0; 1); (0; 1); (0; 0)g (\text{by (b)})
\]

\[
g_{G_3} f(1; 1); (0; 1); (0; 0); \cdots; (0; 0)g (\text{by (d)})
\]

\[
g_{G_3} f(0; 0); (1; 1); (0; 0); \cdots; (0; 0)g (\text{by (b)})
\]
If there is a block between the $r$st block and the $(j - 1)$-st block which is not $\langle 0; 0 \rangle$, by the induction hypothesis, the sequence from the $r$st block to the $(j - 1)$-st block is $G_g$-equivalent to $\langle 0; 1 \rangle; \langle 0; 0 \rangle; \langle 0; 0 \rangle; \langle 0; 0 \rangle$ or $\langle 1; 1 \rangle; \langle 0; 0 \rangle; \langle 0; 0 \rangle; \langle 0; 0 \rangle$. In the $r$st case,

$\begin{align*}
\times_{G_g} f(0; 1); (0; 0); (0; 0); (0; 0); & (0; 0); (1; 1); (0; 0) \\
G_g f(0; 1); (0; 1); (0; 1); & (0; 1); (0; 1); (0; 0) \\
G_g f(0; 1); (0; 0); (0; 1); & (0; 0); (0; 0) \\
G_g f(1; 1); (0; 0); (0; 0); & (0; 0); (0; 0) \\
G_g f(1; 1); (1; 0); (0; 0); & (0; 0); (0; 0) \\
G_g f(0; 0); (1; 1); (0; 0); & (0; 0); (0; 0)
\end{align*}$

by the hypothesis.

In the second case,

$\begin{align*}
\times_{G_g} f(1; 1); (0; 0); (0; 0); (0; 0); & (0; 0); (1; 1); (0; 0) \\
G_g f(1; 1); (0; 1); (0; 1); & (0; 1); (0; 1); (0; 0) \\
G_g f(1; 1); (1; 1); (0; 1); & (0; 1); (0; 0) \\
G_g f(1; 1); (0; 0); (0; 1); & (0; 0); (0; 0) \\
G_g f(0; 1); (0; 0); (0; 0); & (0; 0); (0; 0) \\
G_g f(0; 1); (0; 1); (0; 0); & (0; 0); (0; 0)
\end{align*}$

by the hypothesis.

In the third case,

$\begin{align*}
\times_{G_g} f(0; 0); (1; 1); (0; 0); (0; 0); & (0; 0); (1; 1); (0; 0) \\
G_g f(0; 0); (1; 1); (1; 1); & (0; 1); (0; 1); (0; 0) \\
G_g f(0; 0); (0; 1); (0; 0); & (0; 0); (0; 0) \\
G_g f(0; 0); (0; 1); (0; 0); & (0; 0); (0; 0) \\
G_g f(0; 1); (0; 0); (0; 0); & (0; 0); (0; 0) \\
G_g f(0; 1); (0; 0); (0; 0); & (0; 0); (0; 0)
\end{align*}$

by the hypothesis.
By the fact that $T_{f(0;1);(0;0)}^2 (0;0)g = D_2$, $T_{f(1;1);(0;0)}^2 (0;0)g = (X_1)^2$, $T_{f(0;0);(1;1)}^2 (0;0)g = (Y_2)^2$, and Lemma 3.3, Lemma 3.10 is proved.

3.4 The modulo 2 orthogonal group

In this subsection, we assume $g \equiv 3$. As in the previous subsection, let $\text{Sp}(2g; \mathbb{Z}_2)$ be the natural homomorphism. Let $q: H_1( g; \mathbb{Z}_2)! \mathbb{Z}_2$ be the quadratic form associated with the intersection form $\langle . \rangle_2$ of $H_1( g; \mathbb{Z}_2)$ which satisfies $q(x_i) = q(y_i) = 0$ for the basis $x_i; y_i$ of $H_1( g; \mathbb{Z}_2)$ indicated on Figure 4. We denote $O(2g; \mathbb{Z}_2) = \text{Aut}(H_1( g; \mathbb{Z}_2))$. Because of Lemma 3.10, if we show $T_2 e \in O(2g; \mathbb{Z}_2)$, then $T_2$ follows. For any $z \in H_1( g; \mathbb{Z}_2)$ such that $q(z) = 1$, we denote $T_2 = x + (z; x)z$. Then $T_2$ is an element of $O(2g; \mathbb{Z}_2)$, and we call this a $\mathbb{Z}_2$-transvection about $z$. Dieudonne [2] showed the following theorem.

Theorem 3.16 [2, Proposition 14 on p.42] When $g \equiv 3$, $O(2g; \mathbb{Z}_2)$ is generated by $\mathbb{Z}_2$-transvections.

Let $g$ be the set of $z$ of $H_1( g; \mathbb{Z}_2)$ such that $q(z) = 1$. For any elements $z_1$ and $z_2$ of $g$, we denote $z_1 \square z_2 = z_1 + (z_2; z_1)z_2$. Here, we remark that $T_{z_1} = \text{id}$, $T_{z}T_{z_1}T_{z_2} = T_{z_1} \square z_2$ and $z_1 \square z_2 \square z_2 = z_1$. We denote an element $x_1 + y_1 + x_g + y_g$ of $H_1( g; \mathbb{Z}_2)$ by $[(1; 1); (g; g)]$, and call each $(i; i)$ the $i$-th block. $g$ is a set finitely generated by the operation $\square$.

In fact, we have

Lemma 3.17 Under the operation $\square$, $g$ is generated by $x_i + y_i (1 \leq i \leq g)$, $x_i + x_i + x_{i+1} (1 \leq i \leq g-1)$, and $x_i + x_{i+1} + y_{i+1} (1 \leq i \leq g-1)$.

Proof For an element $[(1; 1); (g; g)]$ of $H_1( g; \mathbb{Z}_2)$, let the $j$-th block be the right most block which is $(1; 1)$. When $j \leq 3$, there exist 4 cases of the combination of the $(j-1)$-st block and the $j$-th block: $[(0; 0); (1; 1); (0; 0); (1; 1); (0; 0); (1; 1); (0; 0); (1; 1); (0; 0); (1; 1); (0; 0); (1; 0); (1; 1); (0; 0); (0; 0); (1; 1); (1; 1); (0; 0)]$. In each case, we can reduce $j$ at least 1. In fact,

\[
\begin{align*}
[(0; 0); (1; 1); (0; 0); (1; 0); (0; 0); (1; 1); (0; 0); (1; 1); (0; 0); (1; 0); (1; 1); (0; 0)] & = [\square(x_j - 1 + x_j + y_j) = [(1; 1); (0; 0); (0; 0); (0; 0)] = [(1; 1); (0; 0); (0; 0); (0; 0)] = [(1; 1); (0; 0); (0; 0); (0; 0)] = [(1; 1); (0; 0); (0; 0); (0; 0)];
\end{align*}
\]

When \( j = 2 \), since \( q(((1; 1); \cdots ; (g; g)) = 1 \), there are \( 3 \) cases of combination of the first block and the second block: \([((0; 0); (1; 1); \cdots ); ((1; 0); (1; 1); \cdots )\), or \([(0; 1); (1; 1); \cdots )\). In each case, \( j \) can be reduced to \( 1 \). In fact,

\[
\begin{align*}
((0; 0); (1; 1); ) & : \square(x_1 + y_1 + x_2) = [(1; 1); (0; 1); ] \\
((1; 0); (1; 1); ) & : \square(x_1 + y_1)\square(x_1 + x_2 + y_2) = [(1; 1); (0; 0); ] \\
((0; 1); (1; 1); ) & : \square(x_1 + x_2 + y_2) = [(1; 1); (0; 0); ]
\end{align*}
\]

When \( j = 1 \), if every \( i \)-th \((i; 2)\) block is \((0; 0)\), then it is \( x_1 + y_1 \). If there exist at least one of the \( i \)-th \((i; 2)\) blocks which are \((1; 0)\) or \((0; 1)\), then,

\[
\begin{align*}
[ & ((0; 0); (1; 0); ) : \square(x_{i-1} + x_i + y_i) = [ (1; 0); (0; 1); ] \\
[ & ((1; 0); (0; 0); ) : \square(x_{i-1} + y_{i-1} + x_i) = [ (0; 1); (1; 0); ] \\
[ & ((0; 0); (0; 1); ) : \square(x_{i-1} + x_i + y_i) = [ (1; 0); (0; 1); ] \\
[ & ((0; 1); (0; 0); ) : \square(x_{i-1} + y_{i-1} + x_i) = [ (0; 0); (1; 0); ]
\end{align*}
\]

Therefore, we can alter this to an element, each \( i \)-th \((i; 2)\) block of which is \((1; 0)\) or \((0; 1)\). If the \( i \)-th block of this is \((0; 1)\), then

\[
[ (0; 1); : \square(x_i + y_i) = [ (1; 0); ]
\]

Therefore, it suffices to consider the case when the first block is \((1; 1)\) and other blocks are \((1; 0)\). In this case,

\[
[ (1; 0); : \square(x_{g-1} + y_{g-1} + x_g)\square(x_{g-1} + y_{g-1}) = [ (1; 0); (0; 0); ]
\]

By applying the same operation repeatedly, we get \([(1; 1); (1; 0); \cdots \] as a result.

This lemma and Theorem 3.16 show:

**Corollary 3.18** \( O(2g; \mathbb{Z}_2) \) is generated by \( T_{x_i+y_i} (1 \leq i \leq g) \), \( T_{x_i+y_i+x_{i+1}} (1 \leq i \leq g-1) \), and \( T_{x_i+x_{i+1}+y_{i+1}} (1 \leq i \leq g-1) \). \( \square \)

Since \( G_g \) is a subgroup of \( \text{SP}_g \), \( 2(G_g) = O(2g; \mathbb{Z}_2) \). On the other hand, the fact that \( 2(X_{2i}) = T_{x_i+y_i+x_{i+1}} (1 \leq i \leq g-1) \), \( 2(X_{2i+1}) = T_{x_i+x_{i+1}+y_{i+1}} (1 \leq i \leq g-1) \), \( 2(X_1) = T_{x_1+y_1} \), \( 2(Y_{2j}) = T_{x_j+y_j} (2 \leq j \leq g-1) \), \( 2(X_{2g}) = T_{x_g+y_g} \), and Corollary 3.18, show \( 2(G_g) = O(2g; \mathbb{Z}_2) \). Therefore we proved that, if \( g = 3 \), then \( \text{SP}_g = G_g \).
3.5 Genus 2 case: Reidemeister-Schreier method

Birman and Hilden showed the following Theorem.

Theorem 3.19 [1] \( M_2 \) is generated by \( C_1; C_2; C_3; C_4; C_5 \) and its defining relations are:

1. \( C_i C_j = C_j C_i \), if \( |i - j| = 2 \), \( i; j = 1; 2; 3; 4; 5 \),
2. \( C_i C_{i+1} C_i = C_{i+1} C_i C_{i+1} \), \( i = 1; 2; 3; 4 \),
3. \( (C_1 C_2 C_3 C_4 C_5)^6 = 1 \),
4. \( (C_1 C_2 C_3 C_4 C_5 C_5 C_4 C_3 C_2 C_1)^2 = 1 \),
5. \( C_1 C_2 C_3 C_4 C_5 C_5 C_4 C_3 C_2 C_1 \equiv C_i \), \( i = 1; 2; 3; 4; 5 \),

where \( \equiv \) means "commute with".

We call (1) (2) of the above relations braid relations. We will use the well-known method, called the Reidemeister-Schreier method [9, 2.3], to show \( S P_2 \cong G_2 \). We review (a part of) this method.

Let \( G \) be a group generated by \( n \) finite elements \( g_1; \ldots; g_n \) and \( H \) be a finite index subgroup of \( G \). For two elements \( a, b \) of \( G \), we write \( a \equiv b \mod H \) if there is an element \( h \) of \( H \) such that \( a = h b \). A finite subset \( S \) of \( G \) is called a coset representative system for \( G \mod H \), if, for each elements \( g \) of \( G \), there is only one element \( \overline{g} \) in \( S \) such that \( g \equiv \overline{g} \mod H \). The set \( \{g \cdot \overline{g}^{-1} \mid i = 1; \ldots; m \} \) where \( \equiv \) means "commute with".

For the sake of giving a coset representative system for \( M_2 \) modulo \( S P_2 \), we will draw a graph \( \Gamma \) which represents the action of \( M_2 \) on the quadratic forms of \( H_1(2; \mathbb{Z}_2) \) with Arf invariants 0. Let \( \{1; 2; 3; 4\} \) denote the quadratic form \( q \) of \( H_1(2; \mathbb{Z}_2) \) such that \( q(x_1) = 1, q(x_1) = 2, q(x_2) = 3, q(x_3) = 4 \). Each vertex of \( \Gamma \) corresponds to a quadratic form. For each generator \( C_i \) of \( M_2 \), we denote its action on \( H_1(2; \mathbb{Z}_2) \) by \( (C_i) \). For the quadratic form \( q \) indicated by the symbol \( \{1; 2; 3; 4\} \), let \( 1 = q((C_i) x_1), 2 = q((C_i) y_1), 3 = q((C_i) x_2), \) and \( 4 = q((C_i) y_2) \). Then, we connect two vertices corresponding to \( \{1; 2; 3; 4\} \) respectively, by the edge with the symbol \( C_i \). We remark that this action is a right action. For simplicity, we omit the edge whose ends are the same vertex. As a result, we get a graph \( \Gamma \) as in Figure 23. (Remark: The same graph was in [4, Proof of Lemma 3.1].) In Figure 23, the bold edges form a maximal tree \( T \) of \( \Gamma \). The words \( S = f(1; C_1; C_2; C_3; C_4; C_5; C_1 C_4; C_2 C_4; C_2 C_5; C_2 C_4 C_3 g \), which
correspond to the edge paths beginning from \([0; 0; 0; 0]\) on \(T\), define a coset representative system for \(M_2\) modulo \(SP_2\). For each element \(g\) of \(M_2\), we can give a \(g \in S\) with using this graph. For example, say \(g = C_2 C_4 C_5 C_2\), we follow an edge path assigned to this word which begins from \([0; 0; 0; 0]\) (note that we read words from left to right) then we arrive at the vertex \([0; 0; 1; 0]\). The edge path on \(T\) which begins from \([0; 0; 0; 0]\) and ends at \([0; 0; 1; 0]\) is \(C_4\). Hence, \(C_2 C_4 C_5 C_2 = C_4\). We list in Table 1 the set of generators \(f s_{C_1} s_{C_2}^{-1}\) \(j = 1; \ldots; 5\), \(s \in S\) of \(SP_2\). In Table 1, vertical direction is a coset representative system \(S\), horizontal direction is a set of generators \(f C_1; C_2; C_3; C_4; C_5 g\). We can check this table by Figure 23 and braid relations. For example,

\[
C_2 C_4 C_3 C_1^{-1} = C_2 C_4 C_3 C_1 (C_2 C_4 C_3)^{-1} = C_2 C_4 C_3 C_1 C_3^{-1} C_4^{-1} C_2^{-1} = C_2 C_1 C_2^{-1} = X_1:
\]

This table shows that \(SP_2 \cong G_2\).
Table 1: Generators of $\mathrm{SP}_2$

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<th></th>
<th>$C_1$</th>
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<td>$X_1$</td>
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<td>$D_5$</td>
</tr>
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References


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