THE DESCENT STATISTIC ON 123-AVOIDING PERMUTATIONS

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Abstract. We exploit Krattenthaler’s bijection between 123-avoiding permutations and Dyck paths to determine the Eulerian distribution over the set $S_n(123)$ of 123-avoiding permutations in $S_n$. In particular, we show that the descents of a permutation correspond to valleys and triple ascents of the associated Dyck path. We get the Eulerian numbers of $S_n(123)$ by studying the joint distribution of these two statistics on Dyck paths.

1. Introduction

A permutation $\sigma \in S_n$ avoids a pattern $\tau \in S_k$ if $\sigma$ does not contain a subsequence that is order-isomorphic to $\tau$. The subset of $S_n$ of all permutations avoiding a pattern $\tau$ is denoted by $S_n(\tau)$. Pattern avoiding permutations have been intensively studied in recent years from many points of view (see e.g. [7], [4], [1], and references therein).

In the case $\tau \in S_3$, it has been shown that the cardinality of $S_n(\tau)$ equals the $n$-th Catalan number, for every pattern $\tau$ (see e.g. [3] and [7]), and hence the set $S_n(\tau)$ is in bijection with the set of Dyck paths of semilength $n$. Indeed, the six patterns in $S_3$ are related as follows:

- $321 = 123^{rev}$,  
- $231 = 132^{rev}$,  
- $312 = 132^c$,  
- $213 = (132^c)^{rev}$,

where $rev$ and $c$ denote the usual reverse and complement operations. Hence, in order to determine the distribution of the descent statistic over $S_n(\tau)$, for every $\tau \in S_3$, it is sufficient to examine the distribution of descents over two sets, say $S_n(132)$ and $S_n(123)$.

In both cases, the two bijections due to Krattenthaler [4] (see also [2]) allow to translate the descent statistic into some appropriate statistics on Dyck paths.

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In the case $\tau = 132$, the descents of a permutation are in one-to-one correspondence with the valleys of the associated Dyck path (see [8] and [9]).

In this paper we investigate the case $\tau = 123$. In particular, we exploit Krattenthaler’s map to translate the descents of a permutation $\sigma \in S_n(123)$ into peculiar subconfigurations of the associated Dyck path, namely, valleys and triple ascents.

For that reason, we study the joint distribution of valleys and triple ascents over the set $P_n$ of Dyck paths of semilength $n$, and we give an explicit expression for its trivariate generating function

$$A(x, y, z) = \sum_{n \geq 0} \sum_{D \in P_n} x^n y^v(D) z^{ta(D)} = \sum_{n, p, q \geq 0} a_{n, p, q} x^n y^p z^q,$$

where $v(D)$ denotes the number of valleys in $D$ and $ta(D)$ denotes the number of triple ascents in $D$. This series specializes to some well known generating functions, such as the generating function of Catalan numbers, Motzkin numbers, Narayana numbers, and sequence A092107 in [8] (see also [5]).

2. DYCK PATHS

A Dyck path of semilength $n$ is a lattice path in the integer lattice $\mathbb{N} \times \mathbb{N}$ starting from the origin, consisting of $n$ up-steps $U = (1, 1)$ and $n$ down steps $D = (1, -1)$, never passing below the x-axis.

A return of a Dyck path is a down step ending on the x-axis, not counting the last step of the Dyck path. An irreducible Dyck path is a Dyck path with no return.

We note that a Dyck path $\mathcal{D}$ can be decomposed according to its last return (last return decomposition) into the juxtaposition of a (possibly empty) Dyck path $\mathcal{D}'$ of shorter length and an irreducible Dyck path $\mathcal{D}''$.

For example, the Dyck path $\mathcal{D} = U^5 D^2 U D^4 U D U^3 D U D^3$ decomposes into $\mathcal{D}' \oplus \mathcal{D}''$, where $\mathcal{D}' = U^5 D^2 U D^4 U D$ and $\mathcal{D}'' = U^3 D U D^3$, as shown in Figure 1.

3. KRATTENTHALER’S BIJECTION

In [4], Krattenthaler describes a bijection between the set $S_n(123)$ and the set $P_n$ of Dyck paths of semilength $n$.

Let $\sigma = \sigma(1) \ldots \sigma(n)$ be a 123-avoiding permutation. Recall that a right-to-left maximum of $\sigma$ is an element $\sigma(i)$ which is larger than $\sigma(j)$ for all $j$ with $j > i$ (note that the last entry $\sigma(n)$ is a right-to-left maximum).
maximum). Let $x_s, \ldots, x_1$ be the right-to-left maxima in $\sigma$. Then, we can write
\begin{equation}
\sigma = w_s x_s \ldots w_1 x_1,
\end{equation}
where $w_i$ are (possibly empty) words. Moreover, since $\sigma$ avoids 123, the word $w_s w_{s-1} \ldots w_1$ must be decreasing.

In order to construct the Dyck path $\kappa(\sigma)$ corresponding to $\sigma$, read the decomposition (1) from right to left. Any right-to-left maximum $x_i$ is translated into $x_i - x_{i-1}$ up steps (with the convention $x_0 = 0$) and any subword $w_i$ is translated into $l_i + 1$ down steps, where $l_i$ denotes the number of elements in $w_i$. Then, reflect the constructed path in a vertical line.

For example, the permutation $\sigma = 6 4 7 3 2 5 1$ in $S_7(123)$ corresponds to the path in Figure 2.

4. The des\-cent statistic

We say that a permutation $\sigma$ has a descent at position $i$ if $\sigma(i) > \sigma(i + 1)$. We denote by des($\sigma$) the number of descents of the permutation $\sigma$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{dyck_path.png}
\caption{The last return decomposition of the Dyck path $D = U^5 D^2 U D^4 U D U^3 D U D^3$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{dyck_path_perm.png}
\caption{The Dyck path $\kappa(\sigma)$, with $\sigma = 6 4 7 3 2 5 1$.}
\end{figure}
In this section we determine the generating function

\[ E(x, y) = \sum_{n \geq 0} \sum_{\sigma \in S_n(123)} x^n y^{\text{des}(\sigma)} = \sum_{n \geq 0} \sum_{k \geq 0} e_{n,k} x^n y^k, \]

where \( e_{n,k} \) denotes the number of permutations in \( S_n(123) \) with \( k \) descents.

**Proposition 1.** Let \( \sigma \) be a permutation in \( S_n(123) \), and \( D = \kappa(\sigma) \).

The number of descents of \( \sigma \) is

\[ \text{des}(\sigma) = v(D) + ta(D), \]

where \( v(D) \) is the number of valleys (the number of occurrences of \( DU \)) in \( D \) and \( ta(D) \) is the number of triple ascents (the number of occurrences of \( UUU \)) in \( D \).

**Proof.** Let \( \sigma = w_s x_s \ldots w_1 x_1 \) be a 123-avoiding permutation. The descents of \( \sigma \) occur precisely in the following positions:

1. between two consecutive symbols in the same word \( w_i \) (we have \( l_i - 1 \) of such descents),
2. after every right-to-left maximum \( x_i \), except for the last one.

The proof is completed as soon as we observe that:

1. every word \( w_i \) is mapped into an ascending run of \( \kappa(\sigma) \) of length \( l_i + 1 \). Such an ascending run contains \( l_i - 1 \) triple ascents, these in their turn are in bijection with the descents contained in \( w_i \),
2. every right-to-left maximum \( x_i \) with \( i \geq 2 \) corresponds to a valley in \( \kappa(\sigma) \).

\[ \square \]

The preceding result implies that we can switch our attention from permutations in \( S_n(123) \) with \( k \) descents to Dyck paths of semilength \( n \) with \( k \) valleys and triple ascents. Hence, we study the joint distribution of valleys and triple ascents over \( \mathcal{P}_n \), namely, we analyze the generating function

\[ A(x, y, z) = \sum_{n \geq 0} \sum_{D \in \mathcal{P}_n} x^n y^{v(D)} z^{ta(D)} = \sum_{n,p,q \geq 0} a_{n,p,q} x^n y^p z^q. \]

We determine the relation between the function \( A(x, y, z) \) and the generating function

\[ B(x, y, z) = \sum_{n \geq 0} \sum_{D \in \mathcal{IP}_n} x^n y^{v(D)} z^{ta(D)} = \sum_{n,p,q \geq 0} b_{n,p,q} x^n y^p z^q \]

of the same joint distribution over the set \( \mathcal{IP}_n \) of irreducible Dyck paths in \( \mathcal{P}_n \).
Proposition 2. For every $n > 2$, we have:

\begin{equation}
(2) \quad b_{n,p,q} = a_{n-1,p,q-1} - a_{n-2,p-1,q-1} + a_{n-2,p-1,q}.
\end{equation}

Proof. An irreducible Dyck path of semilength $n$ with $p$ valleys and $q$ triple ascents can be obtained by prepending $U$ and appending $D$ to a Dyck path of semilength $n - 1$ of one of the two following types:

1. a Dyck path with $p$ valleys and $q$ triple ascents, starting with $UD$;
2. a Dyck path with $p$ valleys and $q - 1$ triple ascents, which does not start with $UD$.

We observe that:

1. The paths of the first kind are in bijection with Dyck paths of semilength $n - 2$ with $p - 1$ valleys and $q$ triple ascents, enumerated by $a_{n-2,p-1,q}$.

\begin{figure}
\centering
\includegraphics[width=.8\textwidth]{dyck_path.png}
\caption{The Dyck path $U^2DU^3D^2U^3D^5$ with 2 valleys and 2 triple ascents is obtained by prepending $UD$ to the path $U^3D^2U^3D^4$ with 1 valley and 2 triple ascents, and then elevating.}
\end{figure}

2. In order to enumerate the paths of the second kind we have to subtract from the integer $a_{n-1,p,q-1}$ the number of Dyck paths of semilength $n - 1$ with $p$ valleys and $q - 1$ triple ascents, starting with $UD$. Dyck paths of this kind are in bijection with Dyck paths of semilength $n - 2$ with $p - 1$ valleys and $q - 1$ triple ascents, enumerated by $a_{n-2,p-1,q-1}$.

\hfill \Box

Proposition 3. For every $n > 0$, we have:

\begin{equation}
(3) \quad a_{n,p,q} = b_{n,p,q} + \sum_{i=1}^{n-1} \sum_{j,s \geq 0} b_{i,j,s} a_{n-i,p-j-1,q-s}.
\end{equation}
Figure 4. The Dyck path $U^3D^2U^3D^5$ with 2 valleys and 2 triple ascents is obtained by elevating the path $U^2D^2U^3D^4$ with 2 valleys and 1 triple ascent.

Proof. Let $D$ be a Dyck path of semilength $n$ and consider its last return decomposition $D = D' \oplus D''$. If $D'$ is empty, then $D$ is irreducible. Otherwise, we have

- $v(D) = v(D') + v(D'') + 1$,
- $ta(D) = ta(D') + ta(D'')$.

□

Identities (2) and (3) yield the following relations between the two generating functions $A(x, y, z)$ and $B(x, y, z)$.

**Proposition 4.** We have

1. $B(x, y, z) = (A(x, y, z) - 1)(xz + x^2y - x^2yz) + 1 + x + x^2 - x^2z$

and

2. $A(x, y, z) = B(x, y, z) + y(B(x, y, z) - 1)(A(x, y, z) - 1)$.

Proof. Note that recurrence (2) holds for $n > 2$. This fact gives rise to the correction terms of $x$-degree less than 3 in Formula (4).

Combining Formulae (4) and (5) we obtain the following result.

**Theorem 5.** We have:

$$A(x, y, z) = \frac{1}{2xy(xy - z - xy)} \left( -1 + xy + 2x^2y ight.$$

$$- 2x^2y^2 + xz - 2xyz - 2x^2yz + 2x^2y^2z$$

$$+ \sqrt{1 - 2xy - 4x^2y + x^2y^2 - 2xz + 2x^2yz + x^2z^2} \right).$$

This last result allows us to determine the generating function $E(x, y)$ of the Eulerian distribution over $S_n(123)$. In fact, the previous arguments show that

$$E(x, y) = A(x, y, y).$$
Hence, we obtain the following explicit expression for $E(x, y)$.

**Theorem 6.** We have:

$$E(x, y) = -1 + 2xy + 2x^2y - 2xy^2 - 4x^2y^2 + 2x^2y^3 + \frac{\sqrt{1 - 4xy - 4x^2y + 4x^2y^2}}{2xy^2(xy - 1 - x)}.$$ 

The first values of the sequence $e_{n,d}$ are shown in the following table:

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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td></td>
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<tr>
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<td>1</td>
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<td>56</td>
<td>252</td>
<td>120</td>
<td>1</td>
</tr>
</tbody>
</table>

Needless to say, the series $A(x, y, z)$ specializes to some well known generating functions. In particular, $A(x, 1, 1)$ is the generating function of Catalan numbers, $A(x, 1, 0)$ the generating function of Motzkin numbers, $yA(x, y, 1)$ the generating function of Narayana numbers, and $A(x, 1, z)$ the generating function of seq. A092107 in [8].

**References**


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