Two new conjectures concerning positive Jacobi polynomials sums

DIMITAR K. DIMITROV* & CLINTON A. MERLO†
Universidade Estadual Paulista, Brasil

Abstract. A refinement of a conjecture of Gasper concerning the values of $(\alpha, \beta)$, $-1/2 < \beta < 0$, $-1 < \alpha + \beta < 0$, for which the inequalities
\[ \sum_{k=0}^{n} P_{k}^{(\alpha,\beta)}(x)/P_{k}^{(\beta,\alpha)}(1) \geq 0, \quad -1 \leq x \leq 1, \quad n = 1, 2, \ldots \]
hold, is stated. An algorithm for checking the new conjecture using the package Mathematica is provided. Numerical results in support of the conjecture are given and a possible approach to its proof is sketched.

Keywords and phrases. Jacobi polynomials, positive sums, Bessel functions, discriminant of a polynomial.

1991 Mathematics Subject Classification. Primary 33C45.

1. Introduction

The Jacobi polynomials are defined in terms of the hypergeometric function \( \, _2F_1 \) by
\[ P_{n}^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)n}{n!} \, _2F_1(-n, n + \alpha + \beta + 1; \alpha + 1; (1 - x)/2), \]

*Research supported by Brazilian Science Fundation CNPq under Grant 300645/95-3.
†Research supported by a fellowship of the Brazilian Science Fundation CAPES.
where \((a)_k = \Gamma (a + k) / \Gamma (a)\) is the Pochhammer symbol and
\[
2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}.
\]

Various special cases of the inequalities
\[
S_n^{(\alpha, \beta)}(x) := \sum_{k=0}^{n} P_k^{(\alpha, \beta)}(x)/P_n^{(\beta, \alpha)}(1) \geq 0, \; -1 \leq x \leq 1, \; n = 1, 2, \ldots \quad (1)
\]
have been proved. Fejér [11, 12] was the first to establish inequalities of this form for \(\alpha = 1/2, \beta = -1/2\) and for \(\alpha = \beta = 0\). Fejér conjectured that (1) also hold for \(\alpha = \beta = 1/2\) and this was proved independently by Jackson [16] and Gronwall [15]. Feldheim [13] proved (1) for \(\beta \geq 0\), \(\alpha + \beta \geq -2\). The importance of the latter result is justified by the fact that de Branges [7] used (1) for \(\beta = 0, \alpha = 2, 4, 6, \ldots\), in the final step of his proof of the celebrated Bieberbach conjecture. Gasper [14] proved inequalities (1) for \(\beta \geq -1/2, \alpha + \beta \geq 0\).

Note that Bateman’s integral formula (Bateman [6])
\[
\frac{P_n^{(\alpha - \mu, \beta + \mu)}(x)}{P_n^{(\beta + \mu, \alpha - \mu)}(1)} = \frac{\Gamma (\beta + \mu + 1)}{\Gamma (\beta + 1) \Gamma (\mu)} \int_{-1}^{x} \frac{P_n^{(\alpha, \beta)}(t)}{P_n^{(\beta, \alpha)}(1)} (1 + t)^{\beta} (x - t)^{\mu-1} dt,
\]
which holds for \(\mu > 0\), and \(\beta > -1\), implies the following result.

**Lemma 1.** If the inequalities (1) holds for \((\alpha, \beta)\), they hold for \((\alpha - \mu, \beta + \mu)\), \(\mu > 0\) as well. Hence, if (1) fail for some \((\alpha, \beta)\) they fail for \((\alpha + \mu, \beta - \mu)\), \(\mu > 0\).

On the other hand \(S_1^{(\alpha, \beta)}(x) = (\alpha + \beta + 2) (1 + x) / (2 (\beta + 1))\). Having in mind these observations, the above mentioned results of Askey and Gasper [4] and of Gasper [14] yield: Inequalities (1) hold for \(\alpha \leq 0, \beta \geq \max \{0, -\alpha - 2\}\) and \(\alpha \geq 0, \beta \geq \max \{-1/2, -\alpha\}\), and fail for \(\beta < \max \{-1/2, -\alpha - 2\}\).

In 1993 Askey [3] drew attention to (1) for the rest of the \((\alpha, \beta)\)-plane, namely, for \((\alpha, \beta)\) in the parallelogram \(D_1 = \{-1/2 \leq \beta < 0, -2 \leq \alpha + \beta < 0\}\). It was proved in [10] that (1) fail for \(x = 1\) and for sufficiently large \(n\), if \(|\alpha - 3/2| - 1/2 \leq \beta < 0\). The latter and Bateman’s integral (2) disprove inequalities (1) for the left hand half of \(D_1\) and \(n\) large enough. Thus the only region in the \((\alpha, \beta)\)-plane for which inequalities (1) is still to be proved or disproved is the parallelogram
\[
D = \{(\alpha, \beta) : -1/2 < \beta < 0, -1 \leq \alpha + \beta < 0\}. 
\]
On the other hand, (1) hold for the upper boundary \( \{ \beta = 0, -1 \leq \alpha < 0 \} \) and fail for the lower boundary \( \{ \beta = -1/2, -1/2 \leq \alpha < 1/2 \} \) of \( D \). Hence, by Bateman’s integral, for any \( \theta \in (-1, 0) \) there exists an \((\alpha', \beta') \in D \) with \( \alpha' + \beta' = \theta \) such that (1) holds for \( \{ \alpha + \beta = \theta, \beta \geq \beta' \} \) and fail for \( \{ \alpha + \beta = \theta, \beta < \beta' \} \).

The curve formed by the points \((\alpha', \beta')\) with this property will be denoted by \( \gamma \). Also, denote by \( J_\alpha(x) \) the Bessel function of the first kind with parameter \( \alpha \) and let \( j_{\alpha,2} \) be the second positive zero of \( J_\alpha(x) \). The following conjecture is due to Gasper [14, p. 444].

**Conjecture 1.** The subregion \( \Delta \) of \( D \) for which the inequalities (1) holds is given by

\[
\Delta = \left\{ (\alpha, \beta) \in D : \beta \geq \beta(\alpha), \int_0^{j_{\alpha,2}} t^{-\beta(\alpha)} J_\alpha(t) \, dt = 0 \right\}.
\]

It may be pointed out that Gaspers’s conjecture is equivalent to the statement that

\[
\gamma = \left\{ (\alpha, \beta(\alpha)) \in D : \int_0^{j_{\alpha,2}} t^{-\beta(\alpha)} J_\alpha(t) \, dt = 0 \right\}.
\]

The conjecture is based on the well-known formula (see (1.8) in [3])

\[
\lim_{n \to \infty} \left( \frac{\theta}{n} \right)^{\alpha-\beta+1} \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(\cos(\theta/n))}{P_k^{(\beta,\alpha)}(1)} = 2^\alpha \Gamma(\beta + 1) \int_0^{\theta} t^{-\beta} J_\alpha(t) \, dt, \quad \beta < \alpha + 1,
\]

and on the following theorem.

**Theorem 1.** Let \(-1 < \alpha < 1/2 \) and \( \beta > -1/2 \). Then the inequality

\[
\int_0^\theta t^{-\beta} J_\alpha(t) \, dt \geq 0
\]

holds for any nonnegative \( \theta \) if and only if

\[
\int_0^{j_{\alpha,2}} t^{-\beta} J_\alpha(t) \, dt \geq 0.
\]

The proof of this theorem for \( \alpha \in (-1, -1/2) \) is due to Askey and Steinig [5] and the case \( \alpha \in (-1/2, 1/2) \) was proved by Makai [17].

Very recently Brown, Koumandos and Wang [8, 9] verified Gasper’s conjecture for the case when \((\alpha, \beta)\) lies on the lines \( \alpha = \beta \) or \( \alpha = -1/2 \).

The objective of the present paper is to state a slight refinement of Conjecture 1 and to give numerical evidence of its truth.
2. The new conjecture

For any positive integer \( n \), set
\[
\Delta_n = \left\{ (\alpha, \beta) \in D : S_n^{(\alpha, \beta)}(x) \geq 0 \text{ for } x \in [-1, 1] \right\}.
\]

Then Gasper’s conjecture can be formulated in the equivalent form
\[
\bigcup_{n=1}^{\infty} \Delta_n = \Delta,
\]
where \( \Delta \) is defined by (3).

We state

**Conjecture 2.** For any positive integer \( n \), \( \Delta_{n+1} \subset \Delta_n \).

Denote by \( \gamma_n \) the boundary of \( \Delta_n \) which passes through \( D \):
\[
\gamma_n = \left\{ (\alpha, \beta) \in D : S_n^{(\alpha, \beta)}(x) \geq 0 \text{ for all } x \in [-1, 1] \text{ and every } (\alpha, \beta) \right\}
\]
with \( \alpha + \beta = \alpha_n + \beta_n, \beta \geq \beta_n \), and for some \( x \in [-1, 1] \), \( S_n^{(\alpha, \beta)}(x) < 0 \)
for \( (\alpha, \beta) \) with \( \alpha + \beta = \alpha_n + \beta_n, \beta < \beta_n \).

The curve \( \gamma_n \) is well defined because of Lemma 1.

An equivalent formulation of Conjecture 2 is that \( \gamma_{n+1} \) lies above \( \gamma_n \) for any positive integer \( n \). The latter conjecture implies that of Gasper, because of (4) and Theorem 1.

In the next section we give explicit expressions for \( \Delta_2 \) and \( \Delta_3 \) or, equivalently, for \( \gamma_2 \) and \( \gamma_3 \). In Section 3 an algorithm to trace the curves \( \gamma_n \) is developed. Tables for the curves \( \gamma_n \) for \( n = 4 \) and \( 5 \) are given and the graphs of \( \gamma_n \) for \( n = 2, 3, 4, 5 \) are drawn. In Section 4 we discuss an idea of how Conjecture 2 might be proved.

3. The cases \( n = 2 \) and \( n = 3 \)

In what follows we suppose that \( (\alpha, \beta) \in D \). First we consider the case \( n = 2 \). Straightforward calculations show that
\[
4 (\beta + 1) (\beta + 2) S_2^{(\alpha, \beta)}(x) = a_2 x^2 + 2a_1 x + a_0,
\]
where
\[
a_2 = (\alpha + \beta + 3) (\alpha + \beta + 4),
\]
\[
a_1 = 2 (\alpha + 2) (\alpha + \beta + 3) + (\alpha + \beta + 2) (\beta + 2) - (\alpha + \beta + 3) (\alpha + \beta + 4)
\]
\[= (\alpha + 1) (\alpha + \beta + 4),
\]
\[
a_0 = 2 (\alpha + \beta + 2) (\beta + 2) + 4 (\alpha + 1) (\alpha + 2) + (\alpha + \beta + 3) (\alpha + \beta + 4)
\]
\[= 4 (\alpha + 2) (\alpha + \beta + 3) = \alpha^2 + 3\beta^2 + 3\alpha + 7\beta + 4.
\]
Obviously $S_2^{(\alpha,\beta)}(x)$ is convex and its minimum value is attained at $x_{\text{min}} = -a_1/a_2 = - (\alpha + 1) / (\alpha + \beta + 3)$. Observe that $-1 < x_{\text{min}} < 0$. Hence, $S_2^{(\alpha,\beta)}(x) \geq 0$ for $x \in [-1,1]$ if and only if it is non-negative for any real $x$. Since its leading coefficient is positive, then $S_2^{(\alpha,\beta)}(x)$ is non-negative if and only if its discriminant

$$(\alpha + 1)^2 (\alpha + \beta + 4)^2 - (\alpha + \beta + 3) (\alpha + \beta + 4) \left( \alpha^2 + 3\beta^2 + 3\alpha + 7\beta + 4 \right)$$

is non-positive. Thus,

$$\Delta_2 = \left\{ (\alpha, \beta) \in D : \beta \geq -3\alpha - 10 + \sqrt{9\alpha^2 + 36\alpha + 52} \right\} / 6.$$

The case $n = 3$ may be treated similarly because $S_n^{(\alpha,\beta)}(-1) = 0$ for any odd $n$. Set $u = (x + 1)/2$. Straightforward calculations show in fact that

$$\overline{S}_3^{(\alpha,\beta)}(u) = \frac{S_3^{(\alpha,\beta)}(x)}{u} = b_2 u^2 - 2b_1 u + b_0$$

where

$$b_2 = (\alpha + \beta + 4)(\alpha + \beta + 5)/(\beta + 1)(\beta + 2)(\beta + 3),$$

$$b_1 = (\alpha + \beta + 4)(\alpha + \beta + 6)/(\beta + 1)(\beta + 2),$$

$$b_0 = 2(\alpha + \beta + 4)/(\beta + 1),$$

and we have to characterize the values of $(\alpha, \beta)$ in $D$ for which $\overline{S}_3^{(\alpha,\beta)}(u) \geq 0$ for each $u \in [0,1]$. Since $\overline{S}_3^{(\alpha,\beta)}(u)$ attains its minimum at $u_{\text{min}} = b_1/b_2 = (\beta + 3)/(\alpha + \beta + 5)$ and $u_{\text{min}} \in [0,1]$, then $\overline{S}_3^{(\alpha,\beta)}(u) \geq 0$ for $u \in [0,1]$ and those $(\alpha, \beta)$ for which the discriminant

$$\left( \frac{(\alpha + \beta + 4)(\alpha + \beta + 6)}{(\beta + 1)(\beta + 2)} \right)^2 - 2 \left( \frac{(\alpha + \beta + 4)^2(\alpha + \beta + 5)(\alpha + \beta + 6)}{(\beta + 1)^2(\beta + 2)(\beta + 3)} \right)$$

of $\overline{S}_3^{(\alpha,\beta)}(u)$ is non-negative. Therefore

$$\Delta_3 = \left\{ (\alpha, \beta) \in D : \beta \geq -\alpha - 5 + \sqrt{\alpha^2 + 6\alpha + 17} \right\} / 2.$$
4. An algorithm to find $\Delta_n$

The algorithm for tracing the curves $\gamma_n$ is based on the following simple fact.

**Lemma 2.** If $(\alpha_n, \beta_n) \in \gamma_n$, then there exists $\xi \in (-1, 1)$ for which

$$S_n^{(\alpha_n, \beta_n)}(\xi) = \frac{d}{dx} S_n^{(\alpha_n, \beta_n)}(\xi) = 0.$$  

**Proof.** Assume that for some $(\alpha_n, \beta_n)$ the polynomial $S_n^{(\alpha_n, \beta_n)}(x)$ is positive at the points of local extrema in $(-1, 1)$. Then a continuity argument implies that there exists a neighborhood $U$ of $(\alpha_n, \beta_n)$ such that for every $(\alpha, \beta)$ in $U$ and for every $x \in (-1, 1)$ the polynomial $S_n^{(\alpha, \beta)}(x)$ is positive. The latter contradicts the definition of $\gamma_n$. $\square$

A well known necessary condition for a polynomial $p(x) = \sum_{\nu=0}^{n} a_{\nu} x^{n-\nu}$ to have a double root is stated in the following lemma. We recall that the discriminant $D(p)$ of $p$ is

$$D(p) = a_0^{2n-2} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2,$$

where $x_1, \ldots, x_n$ are the roots (zeros) of $p$.

**Lemma 3.** The discriminant $D(p)$ of the polynomial $p$ can be represented as a $(2n-1) \times (2n-1)$ determinant in the form

$$\frac{a_0 D(p)}{(-1)^{n-1}} = \begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & a_n \\ na_0 & (n-1) a_1 & \cdots & a_{n-1} & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ a_0 & a_1 & \cdots & a_{n-1} & a_n \\ na_0 & (n-1) a_1 & \cdots & a_{n-1} & \cdot \end{vmatrix}.$$  

Moreover, $D(p) = 0$ if and only if $p(x)$ has at least one root of multiplicity at least two.

We refer to [18, Section 1.3.3] and the references therein for the proof of this lemma and for additional information about discriminants.

Lemmas 2 and 3 immediately yield the following result.

**Theorem 2.** Let $S_n^{(\alpha, \beta)}(x) = \sum_{k=0}^{n} a_k (\alpha_n, \beta_n) x^{n-k}$. If $(\alpha_n, \beta_n) \in \gamma_n$, then

$$D(\alpha_n, \beta_n) := D\left(S_n^{(\alpha, \beta)}\right) = 0.$$
The basic steps of the algorithm to construct an approximation to the curve \( \gamma_n \) are:

1. Choose \( k \in \mathbb{N} \).
2. Divide the interval \([-2, 1/2]\) into \( k \) subintervals by the mesh points \( \alpha_n^{(i)} = -2 + 2.5i/k, \ i = 0, k \).
3. For any fixed \( \alpha_n^{(i)} \) find all the solutions \( \beta_n^{(i)}, \ldots, \beta_n^{(p)} \in (-1/2, 0) \) of the equation \( D \left( \alpha_n^{(i)}, \beta \right) = 0 \).
4. Find that \( s, 1 \leq s \leq p, \) for which \( S_n^{(\alpha_n^{(i)}, \beta_n^{(i)})}(x) \geq 0 \) for \( x \in [-1, 1] \) and
   \[
   S_n^{(\alpha_n^{(i)}, \beta_n^{(i)})}(\xi) = \frac{d}{dx}S_n^{(\alpha_n^{(i)}, \beta_n^{(i)})}(\xi) = 0 \text{ for some } \xi \in (-1, 1).
   \]
5. Choose \( \beta_n^{(i)} = \beta_n^{(i)} \).
6. Approximate the data \( \left( \alpha_n^{(i)}, \beta_n^{(i)} \right) \) by a smooth curve.

Table 1 in the next page contains the results of the algorithm for \( n = 4 \) and \( n = 5 \), for \( k = 50 \). The values of \( \beta_4^{(i)} \) and \( \beta_5^{(i)} \) which correspond to \( \alpha_n^{(i)} = \alpha^{(i)} = -2 + 0.05i, \ i = 0, \ldots, 50 \), are:

The graphs of the approximations to the curves \( \gamma_n \) for \( n = 2, 3, 4 \) and 5 are drawn in Figure 1 at the end of the paper.

5. An idea for proving Conjecture 2

The graphs of the curves \( \gamma_2, \gamma_3, \gamma_4 \) and \( \gamma_5 \) show that Conjecture 2 holds for \( n = 2, 3 \) and 4. It is clear that Conjecture 2 would be proved if one proves that \( S_n^{(\alpha, \beta)} \) is nonnegative on \([-1, -1]\) for any \((\alpha, \beta)\) for which \( S_n^{(\alpha, \beta)} \) is nonnegative there. Another possible idea to prove Conjecture 2 is to show that for any \((\alpha_n, \beta_n) \in \gamma_n \) the inequality \( S_n^{(\alpha_n, \beta_n)}(x) \geq 0 \) fails for some \( x \in [-1, 1] \). It turns out that for \( n = 2, 3 \) and 4 such \( x \) exists. Based on the graphs of \( S_n^{(\alpha_n, \beta_n)}(x) \) and \( S_{n+1}^{(\alpha_n, \beta_n)}(x) \) for various \((\alpha_n, \beta_n) \in \gamma_n \) we may state an additional conjecture which implies the truth of Conjecture 2, and thus, of Conjecture 1.

**Conjecture 3.** Let \((\alpha_n, \beta_n) \in \gamma_n \). Then there exists a unique \( \xi_n \in (-1, 1) \) such that

\[
S_n^{(\alpha_n, \beta_n)}(\xi_n) = \frac{d}{dx}S_n^{(\alpha_n, \beta_n)}(\xi_n) = 0.
\]
Moreover, there exist \( \eta'_n \) and \( \eta''_n \) with \(-1 < \xi_n < \eta'_n < \eta''_n < 1\) such that
\[
S_{n+1, m}^{(\alpha_n, \beta_n)} (x) < 0 \quad \text{for} \quad x \in (\eta'_n, \eta''_n).
\]

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \alpha^{(i)} )</th>
<th>( \beta_4^{(i)} )</th>
<th>( \beta_5^{(i)} )</th>
<th>( i )</th>
<th>( \alpha^{(i)} )</th>
<th>( \beta_4^{(i)} )</th>
<th>( \beta_5^{(i)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-2.00</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>-1.95</td>
<td>-0.0124665</td>
<td>-0.0100482</td>
<td>26</td>
<td>-0.70</td>
<td>-0.29347</td>
<td>-0.271235</td>
</tr>
<tr>
<td>2</td>
<td>-1.90</td>
<td>-0.0248627</td>
<td>-0.020186</td>
<td>27</td>
<td>-0.65</td>
<td>-0.303304</td>
<td>-0.281463</td>
</tr>
<tr>
<td>3</td>
<td>-1.85</td>
<td>-0.0371837</td>
<td>-0.0304035</td>
<td>28</td>
<td>-0.60</td>
<td>-0.313026</td>
<td>-0.291642</td>
</tr>
<tr>
<td>4</td>
<td>-1.80</td>
<td>-0.0494251</td>
<td>-0.0406914</td>
<td>29</td>
<td>-0.55</td>
<td>-0.322637</td>
<td>-0.30177</td>
</tr>
<tr>
<td>5</td>
<td>-1.75</td>
<td>-0.0615829</td>
<td>-0.051041</td>
<td>30</td>
<td>-0.50</td>
<td>-0.332137</td>
<td>-0.311845</td>
</tr>
<tr>
<td>6</td>
<td>-1.70</td>
<td>-0.0736534</td>
<td>-0.0614439</td>
<td>31</td>
<td>-0.45</td>
<td>-0.341526</td>
<td>-0.321856</td>
</tr>
<tr>
<td>7</td>
<td>-1.65</td>
<td>-0.0856334</td>
<td>-0.0718924</td>
<td>32</td>
<td>-0.40</td>
<td>-0.350807</td>
<td>-0.331828</td>
</tr>
<tr>
<td>8</td>
<td>-1.60</td>
<td>-0.0975197</td>
<td>-0.0823791</td>
<td>33</td>
<td>-0.35</td>
<td>-0.359997</td>
<td>-0.341732</td>
</tr>
<tr>
<td>9</td>
<td>-1.55</td>
<td>-0.109331</td>
<td>-0.0928969</td>
<td>34</td>
<td>-0.30</td>
<td>-0.36904</td>
<td>-0.351576</td>
</tr>
<tr>
<td>10</td>
<td>-1.50</td>
<td>-0.121001</td>
<td>-0.103439</td>
<td>35</td>
<td>-0.25</td>
<td>-0.377995</td>
<td>-0.361359</td>
</tr>
<tr>
<td>11</td>
<td>-1.45</td>
<td>-0.132592</td>
<td>-0.1114</td>
<td>36</td>
<td>-0.20</td>
<td>-0.386843</td>
<td>-0.371079</td>
</tr>
<tr>
<td>12</td>
<td>-1.40</td>
<td>-0.144079</td>
<td>-0.124573</td>
<td>37</td>
<td>-0.15</td>
<td>-0.395585</td>
<td>-0.380734</td>
</tr>
<tr>
<td>13</td>
<td>-1.35</td>
<td>-0.155462</td>
<td>-0.135135</td>
<td>38</td>
<td>-0.10</td>
<td>-0.404222</td>
<td>-0.390324</td>
</tr>
<tr>
<td>14</td>
<td>-1.30</td>
<td>-0.166739</td>
<td>-0.145734</td>
<td>39</td>
<td>-0.05</td>
<td>-0.412754</td>
<td>-0.399847</td>
</tr>
<tr>
<td>15</td>
<td>-1.25</td>
<td>-0.177909</td>
<td>-0.156312</td>
<td>40</td>
<td>0.00</td>
<td>-0.421183</td>
<td>-0.409303</td>
</tr>
<tr>
<td>16</td>
<td>-1.20</td>
<td>-0.18897</td>
<td>-0.166881</td>
<td>41</td>
<td>0.05</td>
<td>-0.429509</td>
<td>-0.418691</td>
</tr>
<tr>
<td>17</td>
<td>-1.15</td>
<td>-0.199922</td>
<td>-0.177438</td>
<td>42</td>
<td>0.10</td>
<td>-0.437734</td>
<td>-0.428009</td>
</tr>
<tr>
<td>18</td>
<td>-1.10</td>
<td>-0.210763</td>
<td>-0.110763</td>
<td>43</td>
<td>0.15</td>
<td>-0.445858</td>
<td>-0.437258</td>
</tr>
<tr>
<td>19</td>
<td>-1.05</td>
<td>-0.221493</td>
<td>-0.198469</td>
<td>44</td>
<td>0.20</td>
<td>-0.453883</td>
<td>-0.446436</td>
</tr>
<tr>
<td>20</td>
<td>-1.00</td>
<td>-0.232112</td>
<td>-0.208998</td>
<td>45</td>
<td>0.25</td>
<td>-0.46181</td>
<td>-0.455544</td>
</tr>
<tr>
<td>21</td>
<td>-0.95</td>
<td>-0.242619</td>
<td>-0.219454</td>
<td>46</td>
<td>0.30</td>
<td>-0.469638</td>
<td>-0.464579</td>
</tr>
<tr>
<td>22</td>
<td>-0.90</td>
<td>-0.253014</td>
<td>-0.229886</td>
<td>47</td>
<td>0.35</td>
<td>-0.477371</td>
<td>-0.473543</td>
</tr>
<tr>
<td>23</td>
<td>-0.85</td>
<td>-0.263296</td>
<td>-0.240284</td>
<td>48</td>
<td>0.40</td>
<td>-0.485008</td>
<td>-0.482435</td>
</tr>
<tr>
<td>24</td>
<td>-0.80</td>
<td>-0.273467</td>
<td>-0.250643</td>
<td>49</td>
<td>0.45</td>
<td>-0.49225</td>
<td>-0.491254</td>
</tr>
<tr>
<td>25</td>
<td>-0.75</td>
<td>-0.283524</td>
<td>-0.260961</td>
<td>50</td>
<td>0.50</td>
<td>-0.5</td>
<td>-0.5</td>
</tr>
</tbody>
</table>

**Table 1.** The curves \( \gamma_4 \) and \( \gamma_5 \)
Finally, we recall that Askey [3] conjectured that $\beta(n)$ defined by (3) is a convex function, which is equivalent to assert that the curve $\gamma$ is convex. It seems that every $\gamma_n$ is a convex curve. If so, obviously $\gamma$ would also be convex.

**Figure 1.** The curves $\gamma_2$, $\gamma_3$, $\gamma_4$ and $\gamma_5$.

**References**


[15] T. H. Gronwall, Über die Gibbssche Erscheinung und die trigonometrischen Summen \( \sin x + \frac{1}{2} \sin 2x + \cdots + \frac{1}{n} \sin nx \), Math. Ann. 72 (1912), 228-243.


(Recibido en febrero de 1998; revisado por los autores en septiembre de 1998)