SET-THEORETIC RELATIONS AND BCH-ALGEBRAS WITH TRIVIAL STRUCTURE

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Abstract

In any BCH-algebra we can define a natural relation which is reflexive and anti-symmetric. This relation induces fundamental properties of a BCH-algebra, but not induces the BCH-operation in general. Moreover, some types of BCH-algebras may be obtained from other reflexive and anti-symmetric relations. We describe connections between such relations. We give also some methods of constructions of BCH-algebras from given relations.

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1. Introduction

In 1966 Y. Imai and K. Iséki [6], defined a class of algebras of $(2,0)$-type, called $BCK$-algebras, which, on the one hand, generalizes the notion of the algebra of sets with the set subtraction as the fundamental non-nullary operation, and on the other hand the notion of the implication algebra [7]. $BCK$-algebras have many interesting generalizations such as $BCI$-algebras, $BCC$-algebras and $BCH$-algebras. Any such algebra has a certain natural order induced by its fundamental operation. Such order induces some properties of this operation, but this operation is not induced by this order in general. Moreover, such $BCH$-algebra may also be obtained from some other order. In this note we describe the connection between relations which create a $BCH$-algebra $G$ and the natural order of $G$.

2. Orders and BCH-algebras

By an algebra $(G,·,0)$ we mean a nonempty set $G$ together with a binary multiplication (denoted by juxtaposition) and a certain distinguished element $0$. Such algebra is called a $BCH$-algebra (or $CI$-algebra [1]) if the following conditions hold:

1. $x·0 = 0$,  
2. $(xy)z = (xz)y$,  
3. $xy = yz = 0$ implies $x = y$.

One can prove (cf. [3], [4], [5]) that every BCH-algebra satisfies

4. $x0 = x$,  
5. $0(xy) = (0x)(0y)$.

A BCH-algebra satisfying

6. $((xy)(xz))(yz) = 0$

is called a $BCI$-algebra. A BCH-algebra is called proper (cf. [5]) if it is not a BCI-algebra, i.e. if it does not satisfy (6).
On any BCH-algebra \((G, \cdot, 0)\) one can define the so-called **natural order** by putting

\[
(x \leq y) \text{ if } \forall z \in G (xz = y).
\]

This "order" is a reflexive and anti-symmetric relation, but, in general, it is not transitive.

**Example 1.** It is easily seen that \(G = \{0, a, b, c\} \) with the multiplication defined by the table

\[
\begin{array}{c|cccc}
  & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 0 & 0 \\
b & b & b & 0 & 0 \\
c & c & c & b & 0 \\
\end{array}
\]

is a BCH-algebra. It is a proper because \((ac)(bc) = (bc) \neq 0\). Its natural order is not transitive because \(a \leq b \) and \(b \leq c \) but not \(a \leq c\).

If the natural order of a BCH-algebra \((G, \cdot, 0)\) has 0 as the smallest element, then \((G, \cdot, 0)\) is called a **BCH\(_0\)-algebra**. In other words, a BCH\(_0\)-algebra is a BCH-algebra \((G, \cdot, 0)\) in which

\[
0 \cdot c = 0
\]

holds for all \(c \in G\). A BCH-algebra satisfying (8) is called a **BCK-algebra**.

The natural order of a BCK-algebra \((G, \cdot, 0)\) is a partial order on \(G\) with 0 as smallest element (cf. [7]). Moreover, any BCK-algebra \((G, \cdot, 0)\) may be considered (cf. [7]) as a groupoid \((G, \cdot, 0)\) with the natural order satisfying conditions: \(0 \leq x\), \((xy)(xz) \leq yz\), \(x0 = x\), \(x \leq y \leq z\) imply \(x = y\). Also any BCK-algebra is partially ordered by such natural order, but in this case 0 is not the smallest element in general.

On every set \(G\) equipped with a distinguished element 0 and a relation \(\rho\) we can define a binary multiplication \(\cdot\) in the following way

\[
x \cdot y = \begin{cases} 0 & \text{if } xy \\ x & \text{otherwise} \end{cases}
\]

We say that such algebra has a **trivial structure**. It is clear that any reflexive and anti-symmetric relation \(\rho\) yields a BCH\(_0\)-algebra. Any partial order on \(G\) with 0 as the smallest element defines on \(G\) the structure of a BCK-algebra.

**Proposition 1.** If a BCH\(_2\)-algebra \(G\) has a trivial structure obtained from the reflexive and anti-symmetric relation \(\rho\), then its natural order coincides
with \( \rho \) only in the case when \( \rho \) satisfies the minimum condition, i.e. if \( 0 \rho z \) for every \( z \in G \).

**Proof.** If \( z \leq y \) then \( xy = 0 \). This implies \( xpy \), or \( x = 0 \). Since \( 0py \) for all \( y \in G \), then \( x \leq y \) implies \( xpy \), i.e. \( \leq \rho \). Conversely if \( xpy \) then by definition \( xy = 0 \), which gives \( z \leq y \). Thus, \( \rho \subseteq \leq \) and in the consequence \( \rho = \leq \).

**Example 2.** We will give an example where \( \rho \neq \leq \). Let \( G = \{0, a\} \) and let the reflexive and anti-symmetric relation \( \rho \) be given by \( 0\rho 0, a\rho a, \not{0}\rho a \) and \( \not{a}\rho 0 \). Then \( (G, \cdot, \cdot) \) is a BCH\( _{0} \)-algebra with the trivial structure. Its multiplication is given by the following table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
</tr>
</tbody>
</table>

The natural order of \((G, \cdot, 0)\) satisfies \( 0 \leq a \). Hence \( \rho \neq \leq \).

We say that a relation \( \rho \) defined on a set \( G \) with a distinguished element \( 0 \) is locally reflexive if \( 0\rho 0 \), and locally transitive if \( 0\rho y \) and \( y\rho z \) imply \( 0\rho z \).

**Lemma 1.** Any relation satisfying the minimum condition is locally reflexive and locally transitive.

**Proposition 2.** If a relation satisfying the minimum condition induces on \( G \) the trivial structure of a BCH-algebra, then it is reflexive and anti-symmetric, and coincides with the natural order on this BCH-algebra.

**Proof.** Assume that a relation \( \rho \) satisfies the minimum condition and defines on \( G \) a BCH-algebra \((G, \cdot, 0)\). If \( \rho \) is not reflexive, then there exists \( x \in G \) such that \( \not{0}\rho x \). But in this case we have \( x \cdot x = x \) by (9), and \( x \cdot x = 0 \), as \((G, \cdot, 0)\) is a BCH-algebra. Thus \( x = 0 \), which is in contradiction with local reflexivity.

If \( \rho \) is not anti-symmetric, then there exist \( x, y \in G \), \( x \neq y \) such that \( xpy \) and \( ypx \). Hence \( x \cdot y = 0 \) and \( y \cdot x = 0 \) by (9). But this by (7) implies \( x = y \), which gives a contradiction. Thus, any relation satisfying the minimum condition and defining a BCH-algebra must be reflexive and anti-symmetric. By Proposition 1 such relation coincides with the natural order of this BCH-algebra. The proof is complete. \( \Box \)
Corollary 1. If a relation $\rho$ satisfies the minimum condition and induces on $G$ the trivial structure of a BCK-algebra, then it is a partial order on $G$ and coincides with the natural order on this BCK-algebra.

The following example shows that a BCK-algebra may not be reproduced from its natural order.

Example 3. Consider three algebras defined on the set $G = \{0, a, b, c\}$ by the following tables:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
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</table>

The first algebra is a proper BCK-algebra (cf. [5]). Its natural order is linear: $0 \leq c \leq b \leq a$. The BCK-algebra with the trivial structure defined on $G$ by this order is given by the second table. The third algebra is a BCK-algebra obtained from this linear order by the construction given in [7]. It is not difficult to verify that these three algebras have the same natural order but are not isomorphic.

3. Constructions of BCH-algebras

Now we give some methods of constructions of BCH-algebras with the trivial structure from the given BCH-algebras (with the trivial structure). We start with a generalization of the construction obtained for BCK-algebras by H. Yutani [8].

Let $\{G_i\}_{i \in I}$ be a nonempty family of BCH-algebras such that $G_i \cap G_j = \{0\}$ for any distinct $i, j \in I$. In $\bigcup_{i \in I} G_i$ we define a new multiplication identifying it with a multiplication in any $G_i$, and putting $xy = x$ if belongs to distinct $G_i$. Direct computations show that the union $\bigcup_{i \in I} G_i$ is a BCH-algebra. It is called the disjoint union of $\{G_i\}_{i \in I}$ (cf. [3]).

In a general case where $\{G_i\}_{i \in I}$ is an arbitrary nonempty family of BCH-algebras, we consider $\{G_i \times \{i\}\}_{i \in I}$ and identify all $(0, i)$, where 0, is a constant of $G_i$. By identifying each $x_i \in G_i$ with $(x_i, i)$, the assumption of the definition mentioned above is satisfied. Consequently, we can define
the disjoint union of an arbitrary BCH-algebra. Obviously, if all \( G_i \) have the trivial structure, then the disjoint union of \( \{ G_i \}_{i \in I} \) has also the trivial structure. Moreover, as a consequence of Theorem 5 from [3] we obtain

**Proposition 3.** Let \( \{ S_i \}_{i \in I} \) be an indexed family of subsets of a BCH-algebra \( G \) with the trivial structure induced by the relation \( \rho \). If

1. \( G = \bigcup S_i \),
2. \( S_i \cap S_j = \{0\} \) for any \( i \neq j \),
3. \( x \in S_i \) implies \( \{y \in G : yx = 0\} \subseteq S_i \) for any \( i \in I \),

then all \( S_i \) are subalgebras with the trivial structure induced by \( \rho_i = \rho_{S_i} \), and \( G \) is a disjoint union of \( S_i \).

Also the following two constructions are a generalization of the known constructions for BCK-algebras. These constructions may be simply translated (by (9)) for BCH-algebras without the trivial structure.

**Proposition 4.** Let \((G, \cdot, 0)\) be a BCH-algebra with the trivial structure induced by \( \rho \) and let \( a \notin G \). If we extend \( \rho \) to \( G \cup \{a\} \) putting \( ax = 0, \rho xa, \rho(a\bar{x}) \) and \( \rho_{ax}, \rho(xa) \) for all \( x \in G \setminus \{0\} \), then \( \rho \) induces on \( G \cup \{a\} \) a BCH-algebra with the trivial structure. This new BCH-algebra is proper iff \((G, \cdot, 0)\) is proper.

**Proposition 5.** Let \((G, \cdot, 0)\) be a BCH-algebra with the trivial structure induced by \( \rho \) and let \( a \notin G \). If we extend \( \rho \) to \( G \cup \{a\} \) putting \( axa, xpa \), and \( \rho(axa) \) for all \( x \in G \), then \( \rho \) induces on \( G \cup \{a\} \) a BCH-algebra with the trivial structure. This BCH-algebra is proper iff \((G, \cdot, 0)\) is proper.

4. Ideals and congruences

A nonempty subset \( A \) of a BCI-algebra \((G, \cdot, 0)\) is called an ideal iff \((i) 0 \in A, (ii) \forall x, z \in A \implies y \in A \). Obviously, any such ideal is a subalgebra of \( G \) and induced on \( G \) a congruence \( \theta \) defined by \( x \theta y \) if \( xy, yx \in A \). The set \( G/\theta = \{C_x : x \in G\} \), where \( C_x = \{y \in G : y \theta x\} \) with the operation \( C_x \cdot C_y = C_{xy} \) is a BCI-algebra. Unfortunately, this fact is not true for BCH-algebras.

**Example 4.** Let \( G \) be a proper BCH-algebra from Example 2 in [2]. Routine
calculations prove that \( A = \{0, b, d, f\} \) is an ideal of \( G \), but the relation \( \theta \) defined by this ideal is not a congruence because \( cd \) is not an element of \( \theta \). This gives a negative answer to the problem posed in [2]. On the other hand, one can prove that there exist congruences which are not defined by any ideal.

A special role in BCH-algebras play the congruences induced by some endomorphisms. It is not difficult to verify that the kernel of an endomorphism \( \phi \) of a BCH-algebra \((G, \cdot, 0)\), i.e., the set \( \ker \phi = \{ x \in G : \phi(x) = 0 \} \) is an ideal and the relation \( \theta \) defined by \( xy \in \ker \phi \) is a congruence, if \( \phi \) has the form \( \phi(x) = 0x \) (cf. (5)), then \( G/\ker \phi \) and \( \phi(G) \) are isomorphic BCH-algebras (cf. [3]). These algebras are medial quasigroups. All such algebras with the finite set of generators are the direct product of the so-called cyclic BCH-algebras [3]. On the other hand, \( \phi(G) \) is the largest (in the sense of inclusion) \( \varphi \)-semisimple BCH-algebra contained in \( G \). Similarly, \( \{ x \in G : \phi(x) = x \} \) is the largest Boolean group contained in \( G \).

References


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