BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITIONS FOR A LINEAR THIRD-ORDER EQUATION

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We prove the existence and uniqueness of a strong solution for a linear third-order equation with integral boundary conditions. The proof uses energy inequalities and the density of the range of the generated operator.

1. Introduction

In the rectangle $\Omega = [0,1] \times [0,T]$, we consider the equation

$$
\mathcal{E}u = \frac{\partial^3 u}{\partial t^3} + \frac{\partial}{\partial x} \left( a(x,t) \frac{\partial u}{\partial x} \right) = f(x,t),
$$

(1.1a)

with the initial conditions

$$
u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in (0,1),
$$

(1.1b)

the final condition

$$
\frac{\partial^2 u}{\partial t^2}(x,T) = 0, \quad x \in (0,1),
$$

(1.1c)

the Dirichlet condition

$$
u(0,t) = 0, \quad \forall t \in (0,T),
$$

(1.1d)
and the integral condition
\[ \int_0^1 u(x,t) \, dx = 0, \quad \forall t \in (0,T). \] (1.1e)

In addition, we assume that the function \( a(x,t) \) is bounded with
\[ 0 < a_0 \leq a(x,t) \leq a_1, \] (1.2)
and has bounded partial derivatives such that
\[ c_1' \leq \frac{\partial^k a}{\partial t^k}(x,t) \leq c_k, \quad \forall x \in (0,1), \ t \in (0,T), \ k = 1, 3, \] with \( c_1' \geq 0, \) \[ \left| \frac{\partial a}{\partial x}(x,t) \right| \leq b_1, \quad \text{for} \ (x,t) \in \Omega. \] (1.3)

Various problems arising in heat conduction [4, 6, 14, 15], chemical engineering [9], underground water flow [13], thermoelasticity [21], and plasmaphysics [19] can be reduced to the nonlocal problems with integral boundary conditions. This type of boundary value problems has been investigated in [1, 2, 3, 5, 6, 7, 9, 14, 15, 16, 20, 23] for parabolic equations, in [18, 22] for hyperbolic equations, and in [10, 11, 12] for mixed-type equations. The basic tool in [4, 10, 11, 12, 16, 23] is the energy inequality method which, of course, requires appropriate multipliers and functional spaces. In this paper, we extend this method to the study of a linear third-order partial differential equation. This type of problems is encountered in the study of thermal conductivity [17] and microscale heat transfer [8].

2. Preliminaries

In this paper, we prove the existence and uniqueness of a strong solution of problem (1.1). For this, we consider the solution of problem (1.1) as a solution of the operator equation \( Lu = f \), where \( L \) is the operator with domain of definition \( D(L) \) consisting of functions \( u \in E \) such that \( \sqrt{1-x} (\partial^{k+1} u/\partial t^k \partial x) (x,t) \in L^2(\Omega), \ k = 0, 3 \) and \( u \) satisfies conditions (1.1d) and (1.1e). The operator \( L \) is considered from \( E \) to \( F \), where \( E \) is the Banach space of the functions \( u, \ u \in L^2(\Omega) \), with the finite norm
\[ \| u \|_E^2 = \int_\Omega \frac{(1-x)^2}{2} \left\{ \left| \frac{\partial^3 u}{\partial t^3} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right\} \, dx \, dt \]
\[ + \int_\Omega \left( \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right) \, dx \, dt, \] (2.1)
and $F$ is the Hilbert space of the functions $\mathcal{F} = (f, 0, 0, 0)$, $f \in L^2(\Omega)$, with the finite norm

$$\|\mathcal{F}\|_F^2 = \int_{\Omega} (1 - x)^2 |f|^2 dx dt.$$  \hspace{1cm} (2.2)

Then we establish an energy inequality

$$\|u\|_E \leq k\|Lu\|_F, \quad \forall u \in D(L),$$  \hspace{1cm} (2.3)

and we show that the operator $L$ has the closure $\overline{L}$.

**Definition 2.1.** A solution of the operator equation $\overline{L}u = \mathcal{F}$ is called a strong solution of problem (1.1).

Inequality (2.3) can be extended to $u \in D(\overline{L})$, that is,

$$\|u\|_E \leq k\|\overline{L}u\|_F, \quad \forall u \in D(\overline{L}).$$  \hspace{1cm} (2.4)

From this inequality, we obtain the uniqueness of a strong solution, if it exists, and the equality of the sets $R(\overline{L})$ and $\overline{R(L)}$. Thus, to prove the existence of a strong solution of problem (1.1) for any $\mathcal{F} \in F$, it remains to prove that the set $R(L)$ is dense in $F$.

3. An energy inequality and its applications

**Theorem 3.1.** For any function $u \in D(L)$, there exists the a priori estimate

$$\|u\|_E \leq k\|Lu\|_F,$$  \hspace{1cm} (3.1)

where

$$k^2 = \frac{17\exp(ct)[5 + 4(b_1)^2 / (c'_3 - 3cc_2 + 3c^2c'_1 - c^3a_1 - b_1^2)] + 1}{\min(1, a_0^2c'_3 - 3cc_2 + 3c^2c'_1 - c^3a_1 - b_1^2)},$$  \hspace{1cm} (3.2)

with the constant $c$ satisfying

$$\sup_{(x,t) \in \Omega} \left( \frac{1}{a} \frac{\partial a}{\partial t} \right) \leq c < \inf_{(x,t) \in \Omega} \left( \frac{1}{a} \frac{\partial a}{\partial t} + 1 \right),$$

$$c'_3 - 3cc_2 + 3c^2c'_1 - c^3a_1 - (b_1)^2 > 0,$$  \hspace{1cm} (3.3)

$$c_2 - 2cc'_1 + c^2a_1 - c'_1 + ca_1 < 0.$$
Proof. Let

\[ Mu = (1 - x)^2 \frac{\partial^3 u}{\partial t^3} + 2(1 - x) \int_x \frac{\partial^3 u}{\partial t^3}, \]  

(3.4)

where

\[ J_x u = \int_0^x u(\zeta, t) d\zeta. \]  

(3.5)

We consider the quadratic form

\[ \Phi(u, u) = \text{Re} \int_\Omega \exp(-ct) \bar{\mu} \mu dx dt, \]  

(3.6)

with the constant \( c \) satisfying (3.3), obtained by multiplying (1.1a) by \( \exp(-ct) \bar{\mu} \), integrating over \( \Omega \), and taking the real part. Substituting the expression of \( \mu \) in (3.6), we obtain

\[
\text{Re} \int_\Omega \exp(-ct) \bar{\mu} \mu dx dt \\
= \text{Re} \int_\Omega \exp(-ct)(1 - x)^2 \left| \frac{\partial^3 u}{\partial t^3} \right|^2 dx dt \\
+ 2 \text{Re} \int_\Omega \exp(-ct)(1 - x) \frac{\partial^3 u}{\partial t^3} J_x \frac{\partial^3 u}{\partial t^3} dx dt \\
+ \text{Re} \int_\Omega \exp(-ct) \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial u}{\partial x} \right) \bar{\mu} \mu dx dt.
\]  

(3.7)

Integrating the last two terms on the right-hand side by parts with respect to \( x \) in (3.7) and using the Dirichlet condition (1.1d), we obtain

\[
2 \text{Re} \int_0^1 (1 - x) \exp(-ct) \frac{\partial^3 u}{\partial t^3} J_x \frac{\partial^3 \bar{u}}{\partial t^3} dx = \int_0^1 \exp(-ct) \left| J_x \frac{\partial^3 u}{\partial t^3} \right|^2 dx,
\]  

(3.8)

\[
\text{Re} \int_\Omega \exp(-ct) \frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} \right) \bar{\mu} \mu dx dt \\
= -\text{Re} \int_\Omega \exp(-ct)(1 - x)^2 a \frac{\partial u}{\partial x} \frac{\partial^4 \bar{u}}{\partial t^3 \partial x} dx dt \\
- 2 \text{Re} \int_\Omega \exp(-ct) \frac{\partial a}{\partial x} u J_x \frac{\partial^3 \bar{u}}{\partial t^3} dx dt \\
- 2 \text{Re} \int_\Omega \exp(-ct) au \frac{\partial^3 \bar{u}}{\partial t^3} dx dt.
\]  

(3.9)
Integrating each term by parts in (3.9) with respect to \( t \) and using the initial and final conditions (1.1b) and (1.1c), we get

\[
\text{Re} \int_\Omega \exp(-ct) \frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} \right) \bar{M} u \, dx \, dt
\]

\[
= -2 \text{Re} \int_\Omega \exp(-ct) \frac{\partial a}{\partial x} u \bar{J} \frac{\partial^3 \bar{u}}{\partial t^3} \, dx \, dt
\]

\[
+ \int_\Omega \exp(-ct) \left( \frac{\partial^3 a}{\partial t^3} - 3 \frac{\partial^2 a}{\partial t^2} + 3c^2 \frac{\partial a}{\partial t} - c^3 a \right)
\]

\[
\times \left[ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] \, dx \, dt
\]

\[-3 \int_\Omega \exp(-ct) \left( \frac{\partial a}{\partial t} - ca \right) \left[ \frac{(1-x)^2}{2} \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right] \, dx \, dt
\]

\[
+ \int_0^1 \exp(-ct) a \left[ \frac{(1-x)^2}{2} \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right] \left| \frac{\partial u}{\partial x} \right| \bigg|_{T=t} \, dx
\]

\[-\int_0^1 \exp(-ct) \left( \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a \right) \left[ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] \bigg|_{t=T} \, dx
\]

\[
+ \text{Re} \int_0^1 \exp(-ct) \left( \frac{\partial a}{\partial t} - ca \right) \left\{ (1-x)^2 \frac{\partial^2 \bar{u}}{\partial t \partial x} \frac{\partial u}{\partial x} + 2u \frac{\partial \bar{u}}{\partial t} \right\} \bigg|_{T=t} \, dx.
\]

(3.10)

Substituting (3.8) and (3.10) in (3.7) and using conditions (1.2), (1.3), and (3.3), we obtain

\[
\int_\Omega \exp(-ct) (1-x)^2 \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx \, dt
\]

\[
+ \int_\Omega \exp(-ct) \left\{ c'_3 - 3cc_2 + 3c^2 c'_1 - c^3 a_1 - b_1^2 \right\}
\]

\[
\times \left[ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] \, dx \, dt
\]

\[
\leq \text{Re} \int_\Omega \exp(-ct) \xi u \bar{M} \bar{u} \, dx \, dt.
\]

(3.11)

Again, substituting the expression of \( Mu \) in (3.11) and using elementary inequality, we get
Boundary value problem with integral conditions

\[
\int_\Omega \exp(-ct) \left( \frac{(1-x)^2}{2} \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \right) dx dt \\
+ \int_\Omega \exp(-ct) \left\{ c_3' - 3cc_2 + 3c^2 c_1' - c^3 a_1 - b_1^2 \right\} \\
\times \left[ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] dx dt \\
\leq 17 \int_\Omega \exp(-ct)(1-x)^2 |f|^2 dx dt.
\]

(3.12)

By virtue of (1.1a), we have

\[
\int_\Omega a_0 \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \left( \frac{(1-x)^2}{2} \right) dx dt \\
\leq \int_\Omega (1-x)^2 |f|^2 dx dt + \int_\Omega 2(1-x)^2 \left| \frac{\partial^3 u}{\partial t^3} \right|^2 dx dt \\
+ 4 \int_\Omega b_1^2 \left\{ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right\} dx dt.
\]

(3.13)

This last inequality combined with (3.12) yields

\[
\int_\Omega \frac{(1-x)^2}{2} \left| \frac{\partial^3 u}{\partial t^3} \right|^2 dx dt \\
+ \int_\Omega \left( c_3' - 3cc_2 + 3c^2 c_1' - c^3 a_1 - b_1^2 \right) \left\{ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right\} dx dt \\
+ \int_\Omega a_0^2 \frac{(1-x)^2}{2} \left| \frac{\partial^2 u}{\partial x^2} \right|^2 dx dt \\
\leq \left\{ 17 \exp(cT) \left[ 5 + \frac{4b_1^2}{c_3' - 3cc_2 + 3c^2 c_1' - c^3 a_1 - b_1^2} \right] + 1 \right\} \\
\times \int_\Omega (1-x)^2 |f|^2 dx dt.
\]

(3.14)

Thus, this inequality implies

\[
\int_\Omega \frac{(1-x)^2}{2} \left\{ \left| \frac{\partial^3 u}{\partial t^3} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right\} dx dt \\
+ \int_\Omega \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 dx dt \\
\leq k^2 \int_\Omega (1-x)^2 |f|^2 dx dt,
\]

(3.15)
where
\[
k^2 = \frac{17 \exp(cT) \left[ 5 + 4b_1^2 / \left( c_3 - 3cc_2 + 3c^2 c'_1 - c^3 a_1 - b_1^2 \right) \right] + 1}{\min \left( 1, a_0^2, c_3 - 3cc_2 + 3c^2 c'_1 - c^3 a_1 - b_1^2 \right)}.
\] (3.16)

Then,
\[
\|u\|_E \leq k \|Lu\|_F, \quad \forall u \in D(L).
\] (3.17)

Thus, we obtain the desired inequality. \qed

**Lemma 3.2.** The operator \( L \) from \( E \) to \( F \) admits a closure.

**Proof.** Suppose that \( (u_n) \in D(L) \) is a sequence such that
\[
u_n \to 0 \quad \text{in} \quad E, \quad Lu_n \to \varnothing \quad \text{in} \quad F.
\] (3.18)

We need to show that \( \varnothing = 0 \). We introduce the operator
\[
\mathcal{L}_0 v = -(1-x)^2 \frac{\partial^3 v}{\partial t^3} + \frac{\partial}{\partial x} \left\{ a(x,t) \frac{\partial}{\partial x} \left[ (1-x)^2 v \right] \right\},
\] (3.19)

with domain \( D(\mathcal{L}_0) \) consisting of functions \( v \in W^{2,3}_2(\Omega) \) satisfying
\[
v|_{t=0} = 0, \quad \frac{\partial v}{\partial t} \bigg|_{t=0} = 0, \quad \frac{\partial^2 v}{\partial t^2} \bigg|_{t=0} = 0, \quad v|_{x=0} = 0, \quad \frac{\partial v}{\partial x} \bigg|_{x=0} = 0.
\] (3.20)

We note that \( D(\mathcal{L}_0) \) is dense in the Hilbert space obtained by completing \( L^2(\Omega) \) with respect to the norm
\[
\int_\Omega (1-x)^2 |v|^2 \, dx \, dt = \|v\|^2.
\] (3.21)

Since
\[
\int_\Omega (1-x)^2 f\overline{v} \, dx \, dt = \lim_{n \to +\infty} \int_\Omega (1-x)^2 \mathcal{L} u_n \overline{v} \, dx \, dt
\]
\[
= \lim_{n \to +\infty} \int_\Omega u_n \mathcal{L}_0 \overline{v} \, dx \, dt = 0,
\] (3.22)

for any function \( v \in D(\mathcal{L}_0) \), it follows that \( f = 0 \). \qed
Theorem 3.1 is valid for a strong solution, then we have the inequality
\[ \|u\|_E \leq k \|Lu\|_F, \quad \forall u \in D(L). \] (3.23)

Hence we obtain the following corollary.

**Corollary 3.3.** A strong solution of problem (1.1) is unique if it exists, and depends continuously on \( F \).

**Corollary 3.4.** The range \( R(L) \) of the operator \( L \) is closed in \( F \), and \( R(L) = \overline{R(L)} \).

### 4. Solvability of problem (1.1)

To prove the solvability of problem (1.1), it is sufficient to show that \( R(L) \) is dense in \( F \). The proof is based on the following lemma.

**Lemma 4.1.** Suppose that \( a(x,t) \) and its derivatives \( \partial^4 a / \partial t^3 \partial x \) and \( \partial^2 a / \partial t \partial x \) are bounded. Let \( D_0(L) = \{ u \in D(L) : u(x,0) = 0, (\partial u / \partial t)(x,0) = 0, (\partial^2 u / \partial t^2)(x,T) = 0 \} \). If, for \( u \in D_0(L) \) and for some functions \( w \in L^2(\Omega) \),
\[
\int_{\Omega} (1-x)Lu \tilde{w} \, dx \, dt = 0, \tag{4.1}
\]
then \( w = 0 \).

**Proof.** Equality (4.1) can be written as follows:
\[
\int_{\Omega} (1-x)\tilde{w} \frac{\partial^3 u}{\partial t^3} \, dx \, dt = - \int_{\Omega} \frac{\partial}{\partial \xi} \left( a(1-x) \frac{\partial u}{\partial x} \right) \left\{ \tilde{w} - \int_0^x \tilde{w} \frac{\xi}{1-\xi} \, d\xi \right\} \, dx \, dt. \tag{4.2}
\]

For a given \( \omega(x,t) \), we introduce the function \( \nu(x,t) \) such that
\[
\nu(x,t) = \omega(x,t) - \int_0^x \frac{\omega(\xi,t)}{1-\xi} \, d\xi. \tag{4.3}
\]

From (4.3), we conclude that \( \int_0^1 \nu(x,t) \, dx = 0 \), and thus, we have
\[
\int_{\Omega} \frac{\partial^3 u}{\partial t^3} N\nu \, dx \, dt = - \int_{\Omega} A(t)u\tilde{\nu} \, dx \, dt, \tag{4.4}
\]
where \( A(t)u = (\partial / \partial x)(a(1-x)(\partial u / \partial x)) \) and \( N\nu = (1-x)\nu + J\nu. \)
Following [23], we introduce the smoothing operators

\[
J_{\epsilon}^{-1} = \left( I - \epsilon \left( \frac{\partial^3}{\partial t^3} \right) \right)^{-1}, \quad (J_{\epsilon}^{-1})^* = \left( I + \epsilon \left( \frac{\partial^3}{\partial t^3} \right) \right)^{-1},
\]

with respect to \( t \), which provide the solutions of the respective problems

\[
g_{\epsilon} - \epsilon \frac{\partial^3 g_{\epsilon}}{\partial t^3} = g, \quad g_{\epsilon}(0) = 0, \quad \frac{\partial g_{\epsilon}}{\partial t} (0) = 0, \quad \frac{\partial^2 g_{\epsilon}}{\partial t^2} (T) = 0,
\]

\[
g_{\epsilon}^* + \epsilon \frac{\partial^3 g_{\epsilon}^*}{\partial t^3} = g, \quad g_{\epsilon}^*(0) = 0, \quad \frac{\partial g_{\epsilon}^*}{\partial t} (T) = 0, \quad \frac{\partial^2 g_{\epsilon}^*}{\partial t^2} (T) = 0.
\]

We also have the following properties: for any \( g \in L^2(0,T) \), the functions \( J_{\epsilon}^{-1}(g) \), \( (J_{\epsilon}^{-1})^* g \in W^3_2(0,T) \). If \( g \in D(L) \), then \( J_{\epsilon}^{-1}(g) \in D(L) \) and we have

\[
\lim \| (J_{\epsilon}^{-1})^* g - g \|_{L^2[0,T]} = 0 \quad \text{for} \ \epsilon \to 0,
\]

\[
\lim \| (J_{\epsilon}^{-1}) g - g \|_{L^2[0,T]} = 0 \quad \text{for} \ \epsilon \to 0.
\]

Substituting the function \( u \) in (4.4) by the smoothing function \( u_{\epsilon} \) and using the relation

\[
A(t)u_{\epsilon} = J_{\epsilon}^{-1} Au - \epsilon J_{\epsilon}^{-1} \beta_{\epsilon}(t)u_{\epsilon},
\]

where

\[
\beta_{\epsilon}(t)u_{\epsilon} = 3 \left( \frac{\partial^2 A(t)}{\partial t^2} \right) \frac{\partial u_{\epsilon}}{\partial t} + 3 \left( \frac{\partial A(t)}{\partial t} \right) \frac{\partial^2 u_{\epsilon}}{\partial t^2} + \frac{\partial^3 A(t)}{\partial t^3} \frac{\partial^3 u_{\epsilon}}{\partial t^3}
\]

we obtain

\[
- \int_{\Omega} u N \frac{\partial^3 \bar{u}_{\epsilon}^*}{\partial t^3} \ dx \ dt = \int_{\Omega} A(t)u\bar{v}_{\epsilon}^* \ dx \ dt - \epsilon \int_{\Omega} \beta_{\epsilon}(t)u_{\epsilon}\bar{v}_{\epsilon}^* \ dx \ dt.
\]

Passing to the limit, the equality in the relation (4.10) remains true for all functions \( u \in L^2(\Omega) \) such that \((1-x)(\partial u/\partial x), (\partial/\partial x)((1-x)(\partial u/\partial x)) \in L^2(\Omega), \) and satisfying condition (1.1d).
The operator $A(t)$ has a continuous inverse in $L^2(0,1)$ defined by

$$
A^{-1}(t)g = -\int_{0}^{x} \frac{1}{1-\zeta} \frac{1}{a(\zeta,t)} \int_{0}^{\zeta} g(\eta,t) d\eta d\zeta + C(t) \int_{0}^{x} \frac{1}{1-\zeta} \frac{1}{a(\zeta,t)} d\zeta,
$$

where

$$
C(t) = \frac{\int_{0}^{1} (d\zeta/a(\zeta,t)) \int_{0}^{\zeta} g(\eta,t) d\eta}{\int_{0}^{1} (d\zeta/a(\zeta,t))}.
$$

Then, we have $\int_{0}^{1} A^{-1}(t)g dx = 0$, hence the function $u_\varepsilon = (J_\varepsilon)^{-1}u$ can be represented in the form

$$
u_\varepsilon = (J_\varepsilon)^{-1} A^{-1}(t) A(t) u.
$$

Then

$$
B_\varepsilon(t)g = \frac{\partial^4 a}{\partial t^3 \partial x} J_\varepsilon^{-1} \left[ \frac{1}{a(x,t)} \left( \int_{0}^{x} g(\eta,t) d\eta - C(t) \right) \right]
+ \frac{\partial^3 a}{\partial t^3} J_\varepsilon^{-1} \left[ \frac{g}{a} - \frac{a_x}{a^2(x,t)} \left( \int_{0}^{x} g(\eta,t) d\eta - C(t) \right) \right]
+ \frac{3}{\partial t} \frac{\partial^2 a}{\partial t^2 \partial x} \frac{\partial}{\partial t} J_\varepsilon^{-1} \left[ \frac{1}{a(x,t)} \left( \int_{0}^{x} g(\eta,t) d\eta - C(t) \right) \right]
+ \frac{\partial a_x}{\partial t} J_\varepsilon^{-1} \left[ \frac{g}{a} - \frac{a_x}{a^2(x,t)} \left( \int_{0}^{x} g(\eta,t) d\eta - C(t) \right) \right],
$$

The adjoint of $B_\varepsilon(t)$ has the form

$$
B_\varepsilon^*(t) = \frac{1}{a} (J_\varepsilon^{-1})^* \left[ \frac{\partial^3 a}{\partial t^2 \partial x} \right] + \frac{3}{a} (J_\varepsilon^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial a \partial h}{\partial t} \right)
+ (G_\varepsilon h)(x) - \frac{1}{1/a(x,t)} \frac{1}{1/a(\eta,t)} dx \int_{0}^{1} (1/a(\eta,t)) d\eta (G_\varepsilon h)(1).
$$
\[ (G_\varepsilon h)(x) = \int_{0}^{x} \left( -\frac{3}{a(\zeta,t)} (J_\varepsilon^{-1})^* \cdot \frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial \zeta \partial t} \frac{\partial h}{\partial t} \right) \\
+ \frac{3}{a(\zeta,t)} \frac{1}{a^2(\zeta,t)} (J_\varepsilon^{-1})^* \cdot \frac{\partial}{\partial t} \left( \frac{\partial a \partial h}{\partial \zeta \partial t} \right) \\
- \frac{1}{a(\zeta,t)} (J_\varepsilon^{-1})^* \left( \frac{\partial^4 a}{\partial t^3 \partial \zeta} \right) + \frac{\partial a}{\partial \zeta} \frac{1}{a^2(\zeta,t)} (J_\varepsilon^{-1})^* \left( \frac{\partial^3 a}{\partial t^3 \partial \zeta} \right) \right) d\zeta, \] 

(4.16)

Consequently, equality (4.10) becomes

\[ -\int_{\Omega} u N \frac{\partial^3 v_\varepsilon^*}{\partial t^3} d\tau dt = \int_{\Omega} A(t) u h_\varepsilon dx dt, \] 

(4.17)

where \( h_\varepsilon = v_\varepsilon^* - \varepsilon B_\varepsilon^* v_\varepsilon^* \).

The left-hand side of (4.17) is a continuous linear functional of \( u \). Hence the function \( h_\varepsilon \) has the derivatives \((1-x)(\partial h_\varepsilon/\partial x), (\partial/\partial x)((1-x)(\partial h_\varepsilon/\partial x)) \in L^2(\Omega)\) and the following conditions are satisfied: \( h_\varepsilon|_{x=0} = 0, h_\varepsilon|_{x=1} = 0, \) and \((1-x)(\partial h_\varepsilon/\partial x)|_{x=1} = 0.\)

From the equality

\[ (1-x) \frac{\partial h_\varepsilon}{\partial x} = \left[ I - \varepsilon \frac{1}{a} (J_\varepsilon^{-1})^* (\partial^3 a/\partial t^3) \right] (1-x) \frac{\partial v_\varepsilon^*}{\partial x} \\
- 3\varepsilon \frac{1}{a} (J_\varepsilon^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial a}{\partial t} \frac{\partial}{\partial t} (1-x) \frac{\partial v_\varepsilon^*}{\partial x} \right), \] 

(4.18)

and since the operator \((J_\varepsilon^{-1})^*\) is bounded in \( L^2(\Omega) \), for sufficiently small \( \varepsilon \), we have \( \| \varepsilon (1/a)(J_\varepsilon^{-1})^* (\partial^3 a/\partial t^3) \| < 1 \). Hence the operator \( I - \varepsilon (1/a)(J_\varepsilon^{-1})^* (\partial^3 a/\partial t^3) \) has a bounded inverse in \( L^2(\Omega) \). We conclude that \((1-x)(\partial v_\varepsilon^*/\partial x) \in L^2(\Omega)\).

Similarly, we conclude that \( (\partial/\partial x)((1-x)(\partial v_\varepsilon^*/\partial x)) \) exists and belongs to \( L^2(\Omega) \), and the following conditions are satisfied:

\[ v_\varepsilon^*|_{x=0} = 0, \quad v_\varepsilon^*|_{x=1} = 0, \quad (1-x) \frac{\partial v_\varepsilon^*}{\partial x}|_{x=1} = 0. \] 

(4.19)

Substituting \( u = \int_{0}^{t} \int_{\eta}^{\tau} \exp(\varepsilon \tau) v_\varepsilon^*(\tau) d\tau d\zeta d\eta \) in (4.4), where the constant \( c \) satisfies (3.3), we obtain

\[ \int_{\Omega} \exp(\varepsilon \tau) v_\varepsilon^* N \bar{v} dx dt = -\int_{\Omega} A(t) u \bar{v} dx dt. \] 

(4.20)
Using the properties of smoothing operators, we have

\[
\int_{\Omega} \exp(ct)v_\epsilon^* N \bar{\sigma} \, dx \, dt = - \int_{\Omega} A(t)u \overline{v_\epsilon^*} \, dx \, dt - \varepsilon \int_{\Omega} A(t)u \frac{\partial^3 v_\epsilon^*}{\partial t^3} \, dx \, dt,
\]

(4.21)

and from

\[
\varepsilon \text{Re} \int_{\Omega} A(t)u \frac{\partial^3 v_\epsilon^*}{\partial t^3} \, dx \, dt = \varepsilon \int_{\Omega} (1-x)a \frac{\partial u}{\partial x} \frac{\partial}{\partial t} \frac{\partial^3 v_\epsilon^*}{\partial x \partial t^2} \, dx \, dt
\]

\[
= - \varepsilon \text{Re} \int_{\Omega} (1-x) \frac{\partial a}{\partial t} \frac{\partial u}{\partial x} \frac{\partial}{\partial t} \frac{\partial^3 v_\epsilon^*}{\partial x \partial t^2} \, dx \, dt
\]

\[
+ \varepsilon \int_{\Omega} (1-x) \frac{\partial a}{\partial t} \frac{\partial^2 u}{\partial t \partial x} \frac{\partial}{\partial t} \frac{\partial^2 v_\epsilon^*}{\partial x \partial t} \, dx \, dt
\]

\[
+ \varepsilon \int_{\Omega} a \exp(-ct)(1-x) \left| \frac{\partial v_\epsilon^*}{\partial x} \right|^2 \, dx \, dt
\]

\[
+ \varepsilon \text{Re} \int_{\Omega} (1-x) \frac{\partial a}{\partial t} \frac{\partial^2 u}{\partial t \partial x} \frac{\partial^2 v_\epsilon^*}{\partial x^2} \, dx \, dt,
\]

(4.22)

we have

\[
\varepsilon \text{Re} \int_{\Omega} A(t)u \frac{\partial^3 v_\epsilon^*}{\partial t^3} \, dx \, dt
\]

\[
\geq \varepsilon \int_{\Omega} a \exp(+ct)(1-x) \left| \frac{\partial v_\epsilon^*}{\partial x} \right|^2 \, dx \, dt
\]

\[
- \varepsilon \int_{\Omega} (1-x) \frac{1}{4a} \left( \frac{\partial a}{\partial t} \right)^2 \exp(-ct) \left| \frac{\partial^3 u}{\partial t^2 \partial x} \right|^2 \, dx \, dt
\]

\[
- \varepsilon \int_{\Omega} a \exp(+ct)(1-x) \left| \frac{\partial v_\epsilon^*}{\partial x} \right|^2 \, dx \, dt
\]

\[
- \varepsilon \int_{\Omega} \frac{1-x}{2} \left( \frac{\partial a}{\partial t} \right)^2 \exp(-ct) \left| \frac{\partial u}{\partial x} \right|^2 \, dx \, dt
\]

\[
- \varepsilon \int_{\Omega} \exp(+ct) \frac{1-x}{2} \left| \frac{\partial^3 v_\epsilon^*}{\partial t \partial x^2} \right|^2 \, dx \, dt
\]

\[
- \varepsilon \int_{\Omega} \exp(+ct) \frac{1-x}{2} \left| \frac{\partial v_\epsilon^*}{\partial t \partial x} \right|^2 \, dx \, dt
\]

\[
- \varepsilon \int_{\Omega} \frac{1-x}{2} \left( \frac{\partial a}{\partial t} \right)^2 \exp(-ct) \left| \frac{\partial^2 u}{\partial x \partial t} \right|^2 \, dx \, dt.
\]

(4.23)
Integrating the first term on the right-hand side by parts in (4.21), we obtain

\[ \text{Re} \int_\Omega A(t) u \overline{v_t} dx \, dt \]

\[ \geq - \frac{3}{2} \int_\Omega (1-x) \exp(-ct) \left( \frac{\partial a}{\partial t} - ca \right) \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 dx \, dt \]

\[ + \frac{1}{2} \int_0^1 (1-x) \exp(-ct) \left( a - \left| \frac{\partial a}{\partial t} - ca \right| \right) \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 dx \bigg|_{t=T} \]

\[ - \frac{1}{2} \int_0^1 (1-x) \exp(-ct) \left\{ \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a + \left| \frac{\partial a}{\partial t} - ca \right| \right\} \left| \frac{\partial u}{\partial x} \right|^2 dx \bigg|_{t=T} \]

\[ + \frac{1}{2} \int_\Omega (1-x) \exp(-ct) \left\{ \frac{\partial^3 a}{\partial t^3} - 3c \frac{\partial^2 a}{\partial t^2} + 3c^2 \frac{\partial a}{\partial t} - c^3 a \right\} \left| \frac{\partial u}{\partial x} \right|^2 dx \, dt. \]

(4.24)

Combining (4.23) and (4.24), we get

\[ \text{Re} \int_\Omega \exp(ct) v_t^* N \overline{v} dx \, dt \]

\[ \leq \frac{3}{2} \int_\Omega (1-x) \exp(-ct) \left( c_1 - ca_0 \right) \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 dx \, dt \]

\[ - \frac{1}{2} \int_0^1 (1-x) \exp(-ct) \left( a_0 - c'_1 - ca_1 \right) \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 dx \bigg|_{t=T} \]

\[ + \frac{1}{2} \int_0^1 (1-x) \exp(-ct) \left( c_2 - 2c'_1 c - c^2 a_1 - c'_1 + ca_1 \right) \left| \frac{\partial u}{\partial x} \right|^2 dx \bigg|_{t=T} \]

\[ - \frac{1}{2} \int_\Omega (1-x) \exp(-ct) \left( c'_2 - 3c_2 c + 3c^2 c'_1 - c^3 a_1 \right) \left| \frac{\partial u}{\partial x} \right|^2 dx \, dt \]

\[ + \epsilon \left( \int_\Omega (1-x) \exp(-ct) \frac{c_1^2}{4a_0} \left| \frac{\partial^3 u}{\partial t^2 \partial x} \right|^2 dx \, dt \right) \]

\[ + \int_\Omega (1-x) \exp(-ct) \frac{c_1^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 dx \, dt \]

\[ + \int_\Omega \frac{1-x}{2} \exp(ct) \left| \frac{\partial^3 v_t^*}{\partial t^2 \partial x} \right|^2 dx \, dt \]

\[ + \int_\Omega (1-x) \exp(-ct) \frac{c_1^2}{2} \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 dx \, dt \]

\[ + \int_\Omega \frac{1-x}{2} \exp(ct) \left| \frac{\partial^2 v_t^*}{\partial t \partial x} \right|^2 dx \, dt \].

(4.25)
Boundary value problem with integral conditions

Using conditions (3.3) and inequalities (4.23) and (4.24), we obtain

$$\text{Re} \int_\Omega \exp(\epsilon t)vN\overline{v} \, dx \, dt \leq 0, \quad \text{as } \epsilon \to 0.$$  \hspace{1cm} (4.26)

Since $\text{Re} \int_\Omega \exp(\epsilon t)v J_x \overline{v} \, dx \, dt = 0$, then $v = 0$ a.e.

Finally, from the equality $(1 - x)v + J_x v = (1 - x)w$, we conclude $w = 0$. □

**Theorem 4.2.** The range $R(\overline{L})$ of $\overline{L}$ coincides with $F$.

**Proof.** Since $F$ is Hilbert space, then $R(\overline{L}) = F$ if and only if the relation

$$\int_\Omega (1 - x)^2 \Delta u \overline{f} \, dx \, dt = 0,$$  \hspace{1cm} (4.27)

for arbitrary $u \in D_0(L)$ and $\overline{f} \in F$, implies that $f = 0$.

Taking $u \in D_0(L)$ in (4.27) and using Lemma 4.1, we obtain that $w = (1 - x)f = 0$, then $f = 0$. □

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