We prove the existence and uniqueness of a strong solution for a linear third-order equation with integral boundary conditions. The proof uses energy inequalities and the density of the range of the generated operator.

1. Introduction

In the rectangle $\Omega = [0,1] \times [0,T]$, we consider the equation

$$
\mathcal{L}u = \frac{\partial^3 u}{\partial t^3} + \frac{\partial}{\partial x} \left( a(x,t) \frac{\partial u}{\partial x} \right) = f(x,t),
$$

with the initial conditions

$$
u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad x \in (0,1),
$$

the final condition

$$
\frac{\partial^2 u}{\partial t^2}(x,T) = 0, \quad x \in (0,1),
$$

the Dirichlet condition

$$
u(0,t) = 0, \quad \forall t \in (0,T),
$$
Boundary value problem with integral conditions

and the integral condition

\[ \int_0^1 u(x,t) \, dx = 0, \quad \forall t \in (0,T). \quad (1.1e) \]

In addition, we assume that the function \( a(x,t) \) is bounded with

\[ 0 < a_0 \leq a(x,t) \leq a_1, \quad (1.2) \]

and has bounded partial derivatives such that

\[ c'_k \leq \frac{\partial^k a}{\partial t^k}(x,t) \leq c_k, \quad \forall x \in (0,1), \ t \in (0,T), \ k = 1, 3, \text{ with } c'_1 \geq 0, \quad (1.3) \]

\[ \left| \frac{\partial a}{\partial x}(x,t) \right| \leq b_1, \quad \text{for } (x,t) \in \Omega. \]

Various problems arising in heat conduction \([4, 6, 14, 15]\), chemical engineering \([9]\), underground water flow \([13]\), thermoelasticity \([21]\), and plasmaphysics \([19]\) can be reduced to the nonlocal problems with integral boundary conditions. This type of boundary value problems has been investigated in \([1, 2, 3, 5, 6, 7, 9, 14, 15, 16, 20, 23]\) for parabolic equations, in \([18, 22]\) for hyperbolic equations, and in \([10, 11, 12]\) for mixed-type equations. The basic tool in \([4, 10, 11, 12, 16, 23]\) is the energy inequality method which, of course, requires appropriate multipliers and functional spaces. In this paper, we extend this method to the study of a linear third-order partial differential equation. This type of problems is encountered in the study of thermal conductivity \([17]\) and microscale heat transfer \([8]\).

### 2. Preliminaries

In this paper, we prove the existence and uniqueness of a strong solution of problem (1.1). For this, we consider the solution of problem (1.1) as a solution of the operator equation \( Lu = \mathcal{F} \), where \( L \) is the operator with domain of definition \( D(L) \) consisting of functions \( u \in E \) such that \( \sqrt{1-x} (\partial^{k+1} u / \partial t^k \partial x)(x,t) \in L^2(\Omega) \), \( k = 0, 3 \) and \( u \) satisfies conditions (1.1d) and (1.1e). The operator \( L \) is considered from \( E \) to \( F \), where \( E \) is the Banach space of the functions \( u, u \in L^2(\Omega) \), with the finite norm

\[ \| u \|_E^2 = \int_\Omega \frac{(1-x)^2}{2} \left\{ \left| \frac{\partial^3 u}{\partial t^3} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right\} \, dx \, dt + \int_\Omega \left( \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right) \, dx \, dt, \quad (2.1) \]
and $F$ is the Hilbert space of the functions $\mathcal{F} = (f,0,0,0)$, $f \in L^2(\Omega)$, with the finite norm

$$\|\mathcal{F}\|^2_F = \int_{\Omega} (1-x)^2 |f|^2 \, dx \, dt. \tag{2.2}$$

Then we establish an energy inequality

$$\|u\|_E \leq k \|Lu\|_F, \quad \forall u \in D(L), \tag{2.3}$$

and we show that the operator $L$ has the closure $\overline{L}$.

**Definition 2.1.** A solution of the operator equation $\overline{L}u = \mathcal{F}$ is called a strong solution of problem (1.1).

Inequality (2.3) can be extended to $u \in D(\overline{L})$, that is,

$$\|u\|_E \leq k \|\overline{L}u\|_F, \quad \forall u \in D(\overline{L}). \tag{2.4}$$

From this inequality, we obtain the uniqueness of a strong solution, if it exists, and the equality of the sets $R(\overline{L})$ and $R(L)$. Thus, to prove the existence of a strong solution of problem (1.1) for any $\mathcal{F} \in F$, it remains to prove that the set $R(L)$ is dense in $F$.

### 3. An energy inequality and its applications

**Theorem 3.1.** For any function $u \in D(L)$, there exists the a priori estimate

$$\|u\|_E \leq k \|Lu\|_F, \tag{3.1}$$

where

$$k^2 = \frac{17 \exp(ct) [5 + 4(b_1)^2 / (c_1 - 3cc_2 + 3c^2c_1 - c^3a_1 - b_1^2)] + 1}{\min (1, a_0^2 c_3 - 3cc_2 + 3c^2c_1 - c^3a_1 - b_1^2)}, \tag{3.2}$$

with the constant $c$ satisfying

$$\sup_{(x,t) \in \Omega} \left( \frac{1}{a} \frac{\partial a}{\partial t} \right) \leq c < \inf_{(x,t) \in \Omega} \left( \frac{1}{a} \frac{\partial a}{\partial t} + 1 \right), \tag{3.3}\]$$

$$c_3 - 3cc_2 + 3c^2c_1 - c^3a_1 - (b_1)^2 > 0, \tag{3.3}$$

$$c_2 - 2cc_1 + c^2a_1 - c_1 + ca_1 < 0.$$
Boundary value problem with integral conditions

Proof. Let

\[ Mu = (1 - x)^2 \frac{\partial^3 u}{\partial t^3} + 2(1 - x) J_x \frac{\partial^3 u}{\partial t^3}, \quad (3.4) \]

where

\[ J_x u = \int_0^x u(\zeta, t) d\zeta. \quad (3.5) \]

We consider the quadratic form

\[ \Phi(u, u) = \text{Re} \int_{\Omega} \exp(-ct) \mathcal{E} u Mu \, dx \, dt, \quad (3.6) \]

with the constant \( c \) satisfying (3.3), obtained by multiplying (1.1a) by \( \exp(-ct) \overline{Mu} \), integrating over \( \Omega \), and taking the real part. Substituting the expression of \( Mu \) in (3.6), we obtain

\[ \text{Re} \int_{\Omega} \exp(-ct) \mathcal{E} u Mu \, dx \, dt \]
\[ = \text{Re} \int_{\Omega} \exp(-ct)(1 - x)^2 \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx \, dt \]
\[ + 2 \text{Re} \int_{\Omega} \exp(-ct)(1 - x) \frac{\partial^3 u}{\partial t^3} J_x \frac{\partial^3 u}{\partial t^3} \, dx \, dt \]
\[ + \text{Re} \int_{\Omega} \exp(-ct) \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial u}{\partial x} \right) \overline{Mu} \, dx \, dt. \quad (3.7) \]

Integrating the last two terms on the right-hand side by parts with respect to \( x \) in (3.7) and using the Dirichlet condition (1.1d), we obtain

\[ 2 \text{Re} \int_0^1 (1 - x) \exp(-ct) \frac{\partial^3 u}{\partial t^3} J_x \frac{\partial^3 u}{\partial t^3} \, dx = \int_0^1 \exp(-ct) \left| J_x \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx, \quad (3.8) \]
\[ \text{Re} \int_{\Omega} \exp(-ct) \frac{\partial}{\partial x} \left( a(x, t) \frac{\partial u}{\partial x} \right) \overline{Mu} \, dx \, dt \]
\[ = -\text{Re} \int_{\Omega} \exp(-ct)(1 - x)^2 \frac{\partial u}{\partial x} \frac{\partial^4 \overline{u}}{\partial x \partial t^3} \, dx \, dt \]
\[ - 2 \text{Re} \int_{\Omega} \exp(-ct) \frac{\partial a}{\partial x} u J_x \frac{\partial^3 \overline{u}}{\partial t^3} \, dx \, dt \]
\[ - 2 \text{Re} \int_{\Omega} \exp(-ct) a \frac{\partial^3 \overline{u}}{\partial t^3} \, dx \, dt. \quad (3.9) \]
Integrating each term by parts in (3.9) with respect to $t$ and using the initial and final conditions (1.1b) and (1.1c), we get

\[
\text{Re} \int_\Omega \exp(-ct) \frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} \right) \overline{Mu} \, dx \, dt \\
= -2 \text{Re} \int_\Omega \exp(-ct) \frac{\partial a}{\partial x} u f x \frac{\partial^3 u}{\partial t^3} \, dx \, dt \\
+ \int_\Omega \exp(-ct) \left( \frac{\partial^3 a}{\partial t^3} - 3c \frac{\partial^2 a}{\partial t^2} + 3c^2 \frac{\partial a}{\partial t} - c^3 a \right) \\
\times \left[ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] \, dx \, dt \\
- 3 \int_\Omega \exp(-ct) \left( \frac{\partial a}{\partial t} - ca \right) \left[ \frac{(1-x)^2}{2} \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right] \, dx \, dt \\
+ \int_0^1 \exp(-ct) a \left[ \frac{(1-x)^2}{2} \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 + \left| \frac{\partial u}{\partial t} \right|^2 \right] \, dx \bigg|_{T=t} \\
- \int_0^1 \exp(-ct) \left( \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a \right) \left[ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] \, dx \bigg|_{t=T} \\
+ \text{Re} \int_0^1 \exp(-ct) \left( \frac{\partial a}{\partial t} - ca \right) \left\{ (1-x)^2 \frac{\partial^2 u}{\partial t \partial x} \frac{\partial u}{\partial x} + 2u \frac{\partial u}{\partial t} \right\} \bigg|_{t=t} \, dx.
\]

Substituting (3.8) and (3.10) in (3.7) and using conditions (1.2), (1.3), and (3.3), we obtain

\[
\int_\Omega \exp(-ct)(1-x)^2 \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx \, dt \\
+ \int_\Omega \exp(-ct) \left\{ c_3^2 - 3cc_2 + 3c^2c_1 - c^3 a_1 - b_1^2 \right\} \\
\times \left[ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \right] \, dx \, dt \\
\leq \text{Re} \int_\Omega \exp(-ct) \xi u M \overline{u} \, dx \, dt.
\]

Again, substituting the expression of $Mu$ in (3.11) and using elementary inequality, we get
Boundary value problem with integral conditions

\[
\int_{\Omega} \exp(-ct) \frac{(1-x)^2}{2} \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx \, dt \\
+ \int_{\Omega} \exp(-ct) \left\{ c'_3 - 3cc_2 + 3c^2c'_1 - c^3a_1 - b_1^2 \right\}^2 \times \left[ \left( \frac{(1-x)^2}{2} \right| \frac{\partial u}{\partial x} \right| + |u|^2 \right] \, dx \, dt \\
\leq 17 \int_{\Omega} \exp(-ct)(1-x)^2|f|^2 \, dx \, dt.
\] (3.12)

By virtue of (1.1a), we have

\[
\int_{\Omega} \left[ \frac{(1-x)^2}{2} \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right] \, dx \, dt \\
\leq \int_{\Omega} (1-x)^2|f|^2 \, dx \, dt + \int_{\Omega} 2(1-x)^2 \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx \, dt \\
+ 4 \int_{\Omega} b_1^2 \left\{ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right| + |u|^2 \right\} \, dx \, dt.
\] (3.13)

This last inequality combined with (3.12) yields

\[
\int_{\Omega} \frac{(1-x)^2}{2} \left| \frac{\partial^3 u}{\partial t^3} \right|^2 \, dx \, dt \\
+ \int_{\Omega} \left\{ c'_3 - 3cc_2 + 3c^2c'_1 - c^3a_1 - b_1^2 \right\} \left\{ \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right| + |u|^2 \right\} \, dx \, dt \\
+ \int_{\Omega} a_0^2 \frac{(1-x)^2}{2} \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \, dx \, dt \\
\leq \left\{ 17 \exp(cT) \left[ 5 + \frac{4b_1^2}{c'_3 - 3cc_2 + 3c^2c'_1 - c^3a_1 - b_1^2} + 1 \right] \right\} \\
\times \int_{\Omega} (1-x)^2|f|^2 \, dx \, dt.
\] (3.14)

Thus, this inequality implies

\[
\int_{\Omega} \frac{(1-x)^2}{2} \left\{ \left| \frac{\partial^3 u}{\partial t^3} \right|^2 + \left| \frac{\partial^2 u}{\partial x^2} \right|^2 \right\} \, dx \, dt + \int_{\Omega} \frac{(1-x)^2}{2} \left| \frac{\partial u}{\partial x} \right|^2 + |u|^2 \, dx \, dt \\
\leq k^2 \int_{\Omega} (1-x)^2|f|^2 \, dx \, dt,
\] (3.15)
where
\[
k^2 = \frac{17 \exp(cT) \left[ 5 + 4b_1^2 / (c_3' - 3cc_2 + 3c^2c_1' - c^3a_1 - b_1^2) \right] + 1}{\min \left( 1, a_0^2, c_3' - 3cc_2 + 3c^2c_1' - c^3a_1 - b_1^2 \right)}.
\tag{3.16}
\]

Then,
\[
\|u\|_E \leq k \|Lu\|_F, \quad \forall u \in D(L).
\tag{3.17}
\]

Thus, we obtain the desired inequality.

\textbf{Lemma 3.2.} The operator \(L\) from \(E\) to \(F\) admits a closure.

\textit{Proof.} Suppose that \((u_n) \in D(L)\) is a sequence such that
\[
u_n \to 0 \quad \text{in} \quad E, \quad Lu_n \not\to 0 \quad \text{in} \quad F.
\tag{3.18}
\]

We need to show that \(\mathcal{F} = 0\). We introduce the operator
\[
\mathcal{L}_0 v = -(1-x)^2 \frac{\partial^3 v}{\partial t^3} + \frac{\partial}{\partial x} \left\{ a(x,t) \frac{\partial}{\partial x} \left[ (1-x)^2 v \right] \right\},
\tag{3.19}
\]

with domain \(D(\mathcal{L}_0)\) consisting of functions \(v \in W^{2,3}_2(\Omega)\) satisfying
\[
v|_{t=0} = 0, \quad \frac{\partial v}{\partial t}|_{t=0} = 0, \quad \frac{\partial^2 v}{\partial t^2}|_{t=0} = 0, \quad v|_{x=0} = 0, \quad \frac{\partial v}{\partial x}|_{x=0} = 0.
\tag{3.20}
\]

We note that \(D(\mathcal{L}_0)\) is dense in the Hilbert space obtained by completing \(L^2(\Omega)\) with respect to the norm
\[
\int_\Omega (1-x)^2 |v|^2 dx \, dt = \|v\|^2.
\tag{3.21}
\]

Since
\[
\int_\Omega (1-x)^2 f \bar{v} \, dx \, dt = \lim_{n \to +\infty} \int_\Omega (1-x)^2 \mathcal{L}_n \bar{v} \, dx \, dt
\]
\[
= \lim_{n \to +\infty} \int_\Omega u_n \mathcal{L}_0 \bar{v} \, dx \, dt = 0,
\tag{3.22}
\]

for any function \(v \in D(\mathcal{L}_0)\), it follows that \(f = 0\).

\textbf{Lemma 3.2.} The operator \(L\) from \(E\) to \(F\) admits a closure.
Theorem 3.1 is valid for a strong solution, then we have the inequality
\[ \|u\|_E \leq k\|Lu\|_F, \quad \forall u \in D(L). \quad (3.23) \]
Hence we obtain the following corollary.

**Corollary 3.3.** A strong solution of problem (1.1) is unique if it exists, and depends continuously on \( F \).

**Corollary 3.4.** The range \( R(L) \) of the operator \( L \) is closed in \( F \), and \( R(L) = R(L) \).

### 4. Solvability of problem (1.1)

To prove the solvability of problem (1.1), it is sufficient to show that \( R(L) \) is dense in \( F \). The proof is based on the following lemma.

**Lemma 4.1.** Suppose that \( a(x,t) \) and its derivatives \( \partial^4 a / \partial t^3 \partial x \) and \( \partial^2 a / \partial t \partial x \) are bounded. Let \( D_0(L) = \{ u \in D(L) : u(x,0) = 0, (\partial u / \partial t)(x,0) = 0, (\partial^2 u / \partial t^2)(x,T) = 0 \} \). If, for \( u \in D_0(L) \) and for some functions \( w \in L^2(\Omega) \),
\[ \int_{\Omega} (1-x)^L u \bar{w} \, dx \, dt = 0, \quad (4.1) \]
then \( w = 0 \).

**Proof.** Equality (4.1) can be written as follows:
\[ \int_{\Omega} (1-x)\bar{w} \frac{\partial^3 u}{\partial t^3} \, dx \, dt = -\int_{\Omega} \frac{\partial}{\partial x} \left( a(1-x) \frac{\partial u}{\partial x} \right) \left\{ \bar{w} - \int_{0}^{x} \frac{\bar{w}}{1-\xi} \, d\xi \right\} \, dx \, dt. \quad (4.2) \]
For a given \( w(x,t) \), we introduce the function \( v(x,t) \) such that
\[ v(x,t) = w(x,t) - \int_{0}^{x} \frac{w(\xi,t)}{1-\xi} \, d\xi. \quad (4.3) \]
From (4.3), we conclude that \( \int_{0}^{1} v(x,t) \, dx = 0 \), and thus, we have
\[ \int_{\Omega} \frac{\partial^3 u}{\partial t^3} Nv \, dx \, dt = -\int_{\Omega} A(t)u \bar{v} \, dx \, dt, \quad (4.4) \]
where \( A(t)u = (\partial / \partial x)(a(1-x)(\partial u / \partial x)) \) and \( Nv = (1-x)v + Jv \).
Following [23], we introduce the smoothing operators

\[ J_\epsilon^{-1} = \left( I - \epsilon \left( \frac{\partial^3}{\partial t^3} \right) \right)^{-1}, \quad (J_\epsilon^{-1})^* = \left( I + \epsilon \left( \frac{\partial^3}{\partial t^3} \right) \right)^{-1}, \quad (4.5) \]

with respect to \( t \), which provide the solutions of the respective problems

\[
\begin{align*}
    g_\epsilon - \epsilon \frac{\partial^3 g_\epsilon}{\partial t^3} &= g, \quad g_\epsilon(0) = 0, \quad \frac{\partial g_\epsilon}{\partial t}(0) = 0, \quad \frac{\partial^2 g_\epsilon}{\partial t^2}(T) = 0, \\
    g_\epsilon^* + \epsilon \frac{\partial^3 g_\epsilon^*}{\partial t^3} &= g, \quad g_\epsilon^*(0) = 0, \quad \frac{\partial g_\epsilon^*}{\partial t}(T) = 0, \quad \frac{\partial^2 g_\epsilon^*}{\partial t^2}(T) = 0.
\end{align*}
\]

We also have the following properties: for any \( g \in L^2(0,T) \), the functions \( J_\epsilon^{-1}(g) \), \( (J_\epsilon^{-1})^* g \in W^{3,2}(0,T) \). If \( g \in D(L) \), then \( J_\epsilon^{-1}(g) \in D(L) \) and we have

\[
\begin{align*}
    \lim_{\epsilon \to 0} \left\| (J_\epsilon^{-1})^* g - g \right\|_{L^2[0,T]} &= 0, \\
    \lim_{\epsilon \to 0} \left\| (J_\epsilon^{-1}) g - g \right\|_{L^2[0,T]} &= 0.
\end{align*}
\]

Substituting the function \( u \) in (4.4) by the smoothing function \( u_\epsilon \) and using the relation

\[ A(t)u_\epsilon = J_\epsilon^{-1} Au - \epsilon J_\epsilon^{-1} \beta_\epsilon(t)u_\epsilon, \quad (4.8) \]

where

\[
\beta_\epsilon(t)u_\epsilon = 3 \frac{\partial^2 A(t)}{\partial t^2} \frac{\partial u_\epsilon}{\partial t} + 3 \frac{\partial A(t)}{\partial t} \frac{\partial^2 u_\epsilon}{\partial t^2} + \frac{\partial^3 A(t)}{\partial t^3} u_\epsilon,
\]

we obtain

\[
- \int_\Omega u N \frac{\partial^3 \overline{\nabla}_\epsilon^*}{\partial^3 t} dx dt = \int_\Omega A(t)uv_\epsilon^* dx dt - \epsilon \int_\Omega \beta_\epsilon(t)u_\epsilon \overline{v_\epsilon}^* dx dt. \quad (4.10)
\]

Passing to the limit, the equality in the relation (4.10) remains true for all functions \( u \in L^2(\Omega) \) such that \((1-x)(\partial u/\partial x), (\partial/\partial x)((1-x)(\partial u/\partial x)) \in L^2(\Omega)\), and satisfying condition (1.1d).
The operator $A(t)$ has a continuous inverse in $L^2(0,1)$ defined by

$$A^{-1}(t)g = -\int_0^x \frac{1}{1-\zeta} \frac{1}{a(\zeta,t)} \int_0^\zeta g(\eta,t) d\eta d\zeta + C(t) \int_0^x \frac{1}{1-\zeta} \frac{1}{a(\zeta,t)} d\zeta,$$  \hspace{1cm} (4.11)

where

$$C(t) = \frac{\int_0^1 (d\zeta/a(\zeta,t)) \int_0^\zeta g(\eta,t) d\eta}{\int_0^1 (d\zeta/a(\zeta,t))}. \hspace{1cm} (4.12)$$

Then, we have $\int_0^1 A^{-1}(t)g dx = 0$, hence the function $u_\epsilon = (J_\epsilon)^{-1}u$ can be represented in the form

$$u_\epsilon = (J_\epsilon)^{-1} A^{-1}(t) A(t) u.$$  \hspace{1cm} (4.13)

Then

$$B_\epsilon(t)g = \frac{\partial^4 a}{\partial t^3 \partial x} J_\epsilon^{-1} \left[ \frac{1}{a(x,t)} \left( \int_0^x g(\eta,t) d\eta - C(t) \right) \right]$$

$$+ \frac{\partial^3 a}{\partial t^3} J_\epsilon^{-1} \left[ \frac{g}{a} - \frac{a_x}{a^2(x,t)} \left( \int_0^x g(\eta,t) d\eta - C(t) \right) \right]$$

$$+ 3 \frac{\partial}{\partial t} \frac{\partial^2 a}{\partial t^2 \partial x} \frac{\partial}{\partial t} J_\epsilon^{-1} \left[ \frac{1}{a(x,t)} \left( \int_0^x g(\eta,t) d\eta - C(t) \right) \right]$$

$$+ \frac{\partial a}{\partial t} \frac{\partial}{\partial t} J_\epsilon^{-1} \frac{g}{a} - \frac{a_x}{a^2(x,t)} \left( \int_0^x g(\eta,t) d\eta - C(t) \right).$$  \hspace{1cm} (4.14)

The adjoint of $B_\epsilon(t)$ has the form

$$B^*_\epsilon(t) = \frac{1}{a} (J_\epsilon^{-1})^* \left[ \frac{\partial^3 a}{\partial t^3} \frac{h}{a^2} \right] + \frac{3}{a} (J_\epsilon^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial a}{\partial t} \frac{\partial h}{\partial t} \right)$$

$$+ (G\epsilon h)(x) - \frac{\int_0^1 (1/a(\eta,t)) d\eta}{\int_0^1 (1/a(x,t)) dx} (G\epsilon h)(1),$$  \hspace{1cm} (4.15)
where

\[
(G_\varepsilon h)(x) = \int_0^x \left( -\frac{3}{a(\zeta,t)} (J_{\varepsilon}^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial^2 h}{\partial t \partial \zeta} \right) \right.
+ 3 \frac{\partial a}{\partial \zeta} \frac{1}{a^2(\zeta,t)} (J_{\varepsilon}^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial h}{\partial t} \right)
\left. - \frac{1}{a(\zeta,t)} (J_{\varepsilon}^{-1})^* \left( \frac{\partial^4 a}{\partial t^3 \partial \zeta} h \right) + \frac{\partial a}{\partial \zeta} \frac{1}{a^2(\zeta,t)} (J_{\varepsilon}^{-1})^* \left( \frac{\partial^3 a}{\partial t^3} h \right) \right) d\zeta.
\]

Consequently, equality (4.10) becomes

\[
- \int_{\Omega} u_N \frac{\partial^3 v_\varepsilon^*}{\partial \zeta^3} dx dt = \int_{\Omega} A(t) u h_\varepsilon dx dt,
\]

where \(h_\varepsilon = v_\varepsilon^* - \varepsilon B_\varepsilon v_\varepsilon^*\).

The left-hand side of (4.17) is a continuous linear functional of \(u\). Hence the function \(h_\varepsilon\) has the derivatives \((1-x)(\partial h_\varepsilon / \partial x), (\partial / \partial x)((1-x)(\partial h_\varepsilon / \partial x)) \in L^2(\Omega)\) and the following conditions are satisfied: \(h_\varepsilon|_{x=0} = 0, h_\varepsilon|_{x=1} = 0,\) and \((1-x)(\partial h_\varepsilon / \partial x)|_{x=1} = 0\).

From the equality

\[
(1-x) \frac{\partial h_\varepsilon}{\partial x} = \left[ I - \varepsilon \frac{1}{a} (J_{\varepsilon}^{-1})^* \frac{\partial^3 a}{\partial \zeta^3} \right] (1-x) \frac{\partial v_\varepsilon^*}{\partial x} - 3 \varepsilon \frac{1}{a} (J_{\varepsilon}^{-1})^* \frac{\partial}{\partial t} \left( \frac{\partial a}{\partial t} (1-x) \frac{\partial v_\varepsilon^*}{\partial x} \right),
\]

and since the operator \((J_{\varepsilon}^{-1})^*\) is bounded in \(L^2(\Omega)\), for sufficiently small \(\varepsilon\), we have \(\|e(1/a)(J_{\varepsilon}^{-1})^* (\partial^3 a / \partial \zeta^3)\| < 1\). Hence the operator \(I - \varepsilon (1/a)(J_{\varepsilon}^{-1})^* (\partial^3 a / \partial t^3)\) has a bounded inverse in \(L^2(\Omega)\). We conclude that \((1-x)(\partial v_\varepsilon^* / \partial x) \in L^2(\Omega)\).

Similarly, we conclude that \((\partial / \partial x)((1-x)(\partial v_\varepsilon^* / \partial x))\) exists and belongs to \(L^2(\Omega)\), and the following conditions are satisfied:

\[
v_\varepsilon^*|_{x=0} = 0, \quad v_\varepsilon^*|_{x=1} = 0, \quad (1-x) \frac{\partial v_\varepsilon^*}{\partial x}|_{x=1} = 0.
\]

Substituting \(u = \int_0^t \int_0^\eta \int_0^T \exp(c\tau) v_\varepsilon^*(\tau) d\tau d\zeta d\eta\) in (4.4), where the constant \(c\) satisfies (3.3), we obtain

\[
\int_{\Omega} \exp(ct) v_\varepsilon^* N v_\varepsilon^* dx dt = - \int_{\Omega} A(t) u v_\varepsilon^* dx dt.
\]
Using the properties of smoothing operators, we have

\[
\int_{\Omega} \exp(ct) v^*_\varepsilon N \overline{\sigma} dx dt = - \int_{\Omega} A(t) u \overline{v^*_\varepsilon} dx dt - \varepsilon \int_{\Omega} A(t) u \frac{\partial^3 v^*_\varepsilon}{\partial t^3} dx dt,
\]

(4.21)

and from

\[
\varepsilon \text{Re} \int_{\Omega} A(t) u \frac{\partial^3 v^*_\varepsilon}{\partial t^3} dx dt = \varepsilon \int_{\Omega} (1 - x) a \frac{\partial u}{\partial x} \frac{\partial}{\partial t} \frac{\partial^3 v^*_\varepsilon}{\partial t^3} dx dt
\]

\[
= - \varepsilon \text{Re} \int_{\Omega} (1 - x) \frac{\partial a}{\partial t} \frac{\partial u}{\partial x} \frac{\partial}{\partial t} \frac{\partial^3 v^*_\varepsilon}{\partial t^3} dx dt
\]

\[
+ \varepsilon \text{Re} \int_{\Omega} (1 - x) \frac{\partial a}{\partial t} \frac{\partial^2 u}{\partial t \partial x} \frac{\partial}{\partial t} \frac{\partial^2 v^*_\varepsilon}{\partial t \partial x} dx dt
\]

\[
+ \varepsilon \int_{\Omega} a \exp(-ct)(1 - x) \left| \frac{\partial v^*_\varepsilon}{\partial x} \right|^2 dx dt
\]

\[
+ \varepsilon \text{Re} \int_{\Omega} (1 - x) \frac{\partial a}{\partial t} \frac{\partial^2 u}{\partial t \partial x} \frac{\partial}{\partial t} \frac{\partial v^*_\varepsilon}{\partial t} dx dt,
\]

(4.22)

we have

\[
\varepsilon \text{Re} \int_{\Omega} A(t) u \frac{\partial^3 v^*_\varepsilon}{\partial t^3} dx dt
\]

\[
\geq \varepsilon \int_{\Omega} a \exp(+ct)(1 - x) \left| \frac{\partial v^*_\varepsilon}{\partial x} \right|^2 dx dt
\]

\[
- \varepsilon \int_{\Omega} (1 - x) \frac{1}{4a} \left( \frac{\partial a}{\partial t} \right)^2 \exp(-ct) \left| \frac{\partial^3 u}{\partial t^3} \right|^2 dx dt
\]

\[
- \varepsilon \int_{\Omega} a \exp(+ct)(1 - x) \left| \frac{\partial v^*_\varepsilon}{\partial x} \right|^2 dx dt
\]

\[
- \varepsilon \int_{\Omega} \frac{1 - x}{2} \left( \frac{\partial a}{\partial t} \right)^2 \exp(-ct) \left| \frac{\partial u}{\partial t} \right|^2 dx dt
\]

\[
- \varepsilon \int_{\Omega} \exp(+ct) \frac{1 - x}{2} \left| \frac{\partial^2 v^*_\varepsilon}{\partial t^2 \partial x} \right|^2 dx dt
\]

\[
- \varepsilon \int_{\Omega} \exp(+ct) \frac{1 - x}{2} \left| \frac{\partial^3 v^*_\varepsilon}{\partial t^3} \right|^2 dx dt
\]

\[
- \varepsilon \int_{\Omega} \frac{1 - x}{2} \left( \frac{\partial a}{\partial t} \right)^2 \exp(-ct) \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 dx dt,
\]

(4.23)
Integrating the first term on the right-hand side by parts in (4.21), we obtain

\[
\text{Re} \int_{\Omega} A(t)u^* v^*_x dx \, dt \\
\geq -\frac{3}{2} \int_{\Omega} (1-x) \exp(-ct) \left( \frac{\partial a}{\partial t} - ca \right) \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \, dx \, dt \\
+ \frac{1}{2} \int_{0}^{1} (1-x) \exp(-ct) \left( a - \frac{\partial a}{\partial t} - ca \right) \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \, dx \bigg|_{t=T} \\
- \frac{1}{2} \int_{0}^{1} (1-x) \exp(-ct) \left\{ \frac{\partial^2 a}{\partial t^2} - 2c \frac{\partial a}{\partial t} + c^2 a + \frac{\partial a}{\partial t} - ca \right\} \left| \frac{\partial u}{\partial x} \right|^2 \, dx \bigg|_{t=T} \\
+ \frac{1}{2} \int_{\Omega} (1-x) \exp(-ct) \left\{ \frac{\partial^3 a}{\partial t^3} - 3c \frac{\partial^2 a}{\partial t^2} + 3c^2 \frac{\partial a}{\partial t} - c^3 a \right\} \left| \frac{\partial u}{\partial x} \right|^2 \, dx \, dt.
\]

Combining (4.23) and (4.24), we get

\[
\text{Re} \int_{\Omega} \exp(ct) v^*_x N \overline{v} dx \, dt \\
\leq \frac{3}{2} \int_{\Omega} (1-x) \exp(-ct) \left( c_1 - ca_{0} \right) \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \, dx \, dt \\
- \frac{1}{2} \int_{0}^{1} (1-x) \exp(-ct) \left\{ a_{0} - c'_1 - ca_{1} \right\} \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \, dx \bigg|_{t=T} \\
+ \frac{1}{2} \int_{0}^{1} (1-x) \exp(-ct) \left\{ c_2 - 2c'_1 c - c^2 a_{1} - c'_1 + ca_{1} \right\} \left| \frac{\partial u}{\partial x} \right|^2 \bigg|_{t=T} \, dx \\
- \frac{1}{2} \int_{\Omega} (1-x) \exp(-ct) \left\{ c'_2 - 3c_2 c + 3c^2 c'_1 - c^3 a_{1} \right\} \left| \frac{\partial u}{\partial x} \right|^2 \, dx \, dt \\
+ \varepsilon \left( \int_{\Omega} (1-x) \exp(-ct) \frac{c^2_1}{4a_0} \left| \frac{\partial^3 u}{\partial t^2 \partial x} \right|^2 \, dx \, dt \right. \\
\left. + \int_{\Omega} (1-x) \exp(-ct) \frac{c^2_1}{2} \left| \frac{\partial u}{\partial x} \right|^2 \, dx \, dt \right. \\
\left. + \int_{\Omega} \frac{1-x}{2} \exp(ct) \left| \frac{\partial^3 v^*_x}{\partial t^2 \partial x} \right|^2 \, dx \, dt \right. \\
\left. + \int_{\Omega} (1-x) \exp(-ct) \frac{c^2_1}{2} \left| \frac{\partial^2 u}{\partial t \partial x} \right|^2 \, dx \, dt \right. \\
\left. + \int_{\Omega} \frac{1-x}{2} \exp(ct) \left| \frac{\partial^2 v^*_x}{\partial t \partial x} \right|^2 \, dx \, dt \right) \\
\]

(4.25)
Using conditions (3.3) and inequalities (4.23) and (4.24), we obtain

$$\text{Re} \int_{\Omega} \exp(ct)v N\bar{v} \, dx \, dt \leq 0, \quad \text{as } \varepsilon \rightarrow 0. \quad (4.26)$$

Since \( \text{Re} \int_{\Omega} \exp(ct)v J_x \bar{v} \, dx \, dt = 0 \), then \( v = 0 \) a.e.

Finally, from the equality \((1 - x)v + J_x v = (1 - x)w\), we conclude \( w = 0 \).

\[\square\]

**Theorem 4.2.** The range \( R(\bar{L}) \) of \( \bar{L} \) coincides with \( F \).

**Proof.** Since \( F \) is Hilbert space, then \( R(\bar{L}) = F \) if and only if the relation

$$\int_{\Omega} (1 - x)^2 \mathcal{E} u \bar{f} \, dx \, dt = 0, \quad (4.27)$$

for arbitrary \( u \in D_0(L) \) and \( \mathcal{F} \in F \), implies that \( f = 0 \).

Taking \( u \in D_0(L) \) in (4.27) and using Lemma 4.1, we obtain that \( w = (1 - x)f = 0 \), then \( f = 0 \).

\[\square\]

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</tr>
</thead>
<tbody>
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<td>First Round of Reviews</td>
<td>March 1, 2009</td>
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<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
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</tbody>
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