SOME RESULTS ON FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS IN COMPLETE METRIC SPACES

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Using the concept of \( w \)-distance, we improve some well-known fixed point theorems.

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1. Introduction. Recently, Ume [3] improved the fixed point theorems in a complete metric space using the concept of \( w \)-distance, introduced by Kada, Suzuki, and Takahashi [2], and more general contractive mappings than quasi-contractive mappings.

In this paper, using the concept of \( w \)-distance, we first prove common fixed point theorems for multivalued mappings in complete metric spaces, then these theorems are used to improve Cirić’s fixed point theorem [1], Kada-Suzuki-Takahashi’s fixed point theorem [2], and Ume’s fixed point theorem [3].

2. Preliminaries. Throughout, we denote by \( \mathbb{N} \) the set of all positive integers and by \( \mathbb{R} \) the set of all real numbers.

**Definition 2.1** (see [2]). Let \( (X, d) \) be a metric space, then a function \( p : X \times X \to [0, \infty) \) is called a \( w \)-distance on \( X \) if the following are satisfied:

1. \( p(x, z) \leq p(x, y) + p(y, z) \) for all \( x, y, z \in X \);
2. for any \( x \in X \), \( p(x, \cdot) : X \to [0, \infty) \) is lower semicontinuous;
3. for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( p(z, x) \leq \delta \) and \( p(z, y) \leq \delta \) imply \( d(x, y) \leq \epsilon \).

**Definition 2.2**. Let \( (X, d) \) be a metric space with a \( w \)-distance \( p \), then

1. for any \( x \in X \) and \( A \subseteq X \), \( d(x, A) := \inf \{ d(x, y) : y \in A \} \) and \( d(A, x) := \inf \{ d(y, x) : y \in A \} \);
2. for any \( x \in X \) and \( A \subseteq X \), \( p(x, A) := \inf \{ p(x, y) : y \in A \} \) and \( p(A, x) := \inf \{ p(y, x) : y \in A \} \);
3. for any \( A, B \subseteq X \), \( p(A, B) := \inf \{ p(x, y) : x \in A, y \in B \} \);
4. \( CB_p(X) = \{ A \mid A \text{ is nonempty closed subset of } X \text{ and } \sup_{x,y \in A} p(x, y) < \infty \} \).

The following lemmas are fundamental.

**Lemma 2.3** (see [2]). Let \( X \) be a metric space with a \( w \)-distance \( p \), then

1. for any \( x \in X \) and \( A \subseteq X \), \( d(x, A) := \inf \{ d(x, y) : y \in A \} \) and \( d(A, x) := \inf \{ d(y, x) : y \in A \} \);
2. for any \( x \in X \) and \( A \subseteq X \), \( p(x, A) := \inf \{ p(x, y) : y \in A \} \) and \( p(A, x) := \inf \{ p(y, x) : y \in A \} \);
3. for any \( A, B \subseteq X \), \( p(A, B) := \inf \{ p(x, y) : x \in A, y \in B \} \);
4. \( CB_p(X) = \{ A \mid A \text{ is nonempty closed subset of } X \text{ and } \sup_{x,y \in A} p(x, y) < \infty \} \).
(1) if \( p(x_n, y) \leq \alpha_n \) and \( p(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \( y = z \). In particular, if \( p(x, y) = 0 \) and \( p(x, z) = 0 \), then \( y = z \);

(2) if \( p(x_n, y_n) \leq \alpha_n \) and \( p(x_n, z) \leq \beta_n \) for any \( n \in \mathbb{N} \), then \( \{y_n\} \) converges to \( z \);

(3) if \( p(x_n, x_m) \leq \alpha_n \) for any \( n, m \in \mathbb{N} \) with \( m > n \), then \( \{x_n\} \) is a Cauchy sequence;

(4) if \( p(y, x_n) \leq \alpha_n \) for any \( n \in \mathbb{N} \), then \( \{x_n\} \) is a Cauchy sequence.

**Lemma 2.4** (see [3]). Let \( X \) be a metric space with a metric \( d \), let \( p \) be a \( w \)-distance on \( X \), and let \( T \) be a mapping of \( X \) into itself satisfying

\[
p(Tx, Ty) \leq q \cdot \max\{p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx)\}
\]

for all \( x, y \in X \) and some \( q \in [0, 1) \). Then

(1) for each \( x \in X \), \( n \in \mathbb{N} \), and \( i, j \in \mathbb{N} \) with \( i \leq n \),

\[
p(T^i x, T^j x) \leq q \cdot \delta(O(x, n));
\]

(2) for each \( x \in X \) and \( n \in \mathbb{N} \), there exist \( k, l \in \mathbb{N} \) with \( k, l \leq n \) such that

\[
\delta(O(x, n)) = \max\{p(x, y), p(x, T^k x), p(T^l x, x)\};
\]

(3) for each \( x \in X \),

\[
\delta(O(x, \infty)) \leq \frac{1}{1-q} \left\{ p(x, x) + p(x, Tx) + p(Tx, x) \right\};
\]

(4) for each \( x \in X \), \( \{T^n x\}_{n=1}^{\infty} \) is a Cauchy sequence.

3. Main results

**Theorem 3.1**. Let \( X \) be a complete metric space with a metric \( d \) and let \( p \) be a \( w \)-distance on \( X \). Suppose that \( S \) and \( T \) are two mappings of \( X \) into \( CB_p(X) \) and \( \varphi : X \times X \to [0, \infty) \) is a mapping such that

\[
\max\{p(u_1, u_2), p(v_1, v_2)\} \leq q \cdot \varphi(x, y)
\]

for all nonempty subsets \( A, B \) of \( X \), \( u_1 \in SA, u_2 \in S^2 A, v_1 \in TB, v_2 \in T^2 B, x \in A, y \in B \), and some \( q \in [0, 1) \),

\[
\sup \left\{ \sup \left( \frac{\varphi(x, y)}{\min\{p(x, SA), p(y, TB)\}} : x \in A, y \in B \right) : A, B \subseteq X \right\} < \frac{1}{q},
\]

\[
\inf \left\{ p(y, u) + p(x, x) + p(y, Ty) : x, y \in X \right\} > 0,
\]

for every \( u \in X \) with \( u \notin Su \) or \( u \notin Tu \), where \( SA = \bigcup_{a \in A} Sa \). Then \( S \) and \( T \) have a common fixed point in \( X \).

**Proof.** Let

\[
\beta = \sup \left\{ \sup \left( \frac{\varphi(x, y)}{\min\{p(x, SA), p(y, TB)\}} : x \in A, y \in B \right) : A, B \subseteq X \right\},
\]

for every \( u \in X \) with \( u \notin Su \) or \( u \notin Tu \), where \( SA = \bigcup_{a \in A} Sa \). Then \( S \) and \( T \) have a common fixed point in \( X \).
and \( k = \beta q \). Define \( x_{n+1} \in Sx_n \) and \( y_{n+1} \in Ty_n \) for all \( n \in \mathbb{N} \). Then \( x_n \in Sx_{n-1} \), \( x_{n+1} \in S^2x_{n-1} \), \( y_n \in Ty_{n-1} \), and \( y_{n+1} \in T^2y_{n-1} \). From (3.1) and (3.2), we have

\[
p(x_n, x_{n+1}) \leq kp(x_{n-1}, x_n) \leq \cdots \leq k^{n-1}p(x_1, x_2), \quad (3.5)
\]

\[
p(y_n, y_{n+1}) \leq kp(y_{n-1}, y_n) \leq \cdots \leq k^{n-1}p(y_1, y_2), \quad (3.6)
\]

for all \( n \in \mathbb{N} \) and some \( k \in [0,1) \). Let \( n \) and \( m \) be any positive integers such that \( n < m \). Then, from (3.6), we obtain

\[
p(y_n, y_m) \leq p(y_n, y_{n+1}) + \cdots + p(y_{m-1}, y_m)
= \sum_{i=0}^{m-n-1} p(y_{n+i}, y_{n+i+1})
\leq \sum_{i=0}^{m-n-1} k^{n+i-1}p(y_1, y_2)
\leq \frac{k^{n-1}}{(1-k)}p(y_1, y_2).
\]

By Lemma 2.3, \( \{y_n\} \) is a Cauchy sequence. Since \( X \) is complete, \( \{y_n\} \) converges to \( u \in X \). Then, since \( p(y_n, \cdot) \) is lower semicontinuous, from (3.7) we have

\[
p(y_n, u) \leq \lim_{m \to \infty} \inf p(y_n, y_m) \leq \frac{k^{n-1}}{(1-k)}p(y_1, y_2). \quad (3.8)
\]

Suppose that \( u \notin Su \) or \( u \notin Tu \). Then, by (3.3), (3.5), (3.6), and (3.8), we have

\[
0 < \inf \{p(y, u) + p(x, Sx) + p(y, Ty) : x, y \in X\}
\leq \inf \{p(y_n, u) + p(x_n, x_{n+1}) + p(y_n, y_{n+1}) : n \in \mathbb{N}\}
\leq \inf \left\{ \frac{k^{n-1}}{(1-k)}p(y_1, y_2) + k^{n-1}p(x_1, x_2) + k^{n-1}p(y_1, y_2) : n \in \mathbb{N}\right\}
= \frac{2-k}{(1-k)}p(y_1, y_2) + k^{n-1}p(x_1, x_2) + \inf \{k^{n-1} : n \in \mathbb{N}\}
= 0.
\]

This is a contradiction. Therefore we have \( u \in Su \) and \( u \in Tu \). \( \square \)

**Theorem 3.2.** Let \( X \) be a complete metric space with a metric \( d \) and let \( p \) be a \( w \)-distance on \( X \). Suppose that \( S \) and \( T \) are two mappings of \( X \) into \( \text{CB}_p(X) \) and \( \varphi : X \times X \to [0, \infty) \) is a mapping such that

\[
\max \{p(u_1, u_2), p(v_1, v_2)\} \leq q \cdot \varphi(x, y) \quad (3.10)
\]

for all \( x, y \in X, u_1 \in Sx, u_2 \in S^2x, v_1 \in Ty, v_2 \in T^2y, \) and some \( q \in [0,1) \),

\[
\sup \left\{ \sup \left( \frac{\varphi(x, y)}{\min \{p(x, Sx), p(y, Ty)\} : x \in A, y \in B\} : A, B \subseteq X \right) \right\} < \frac{1}{q}, \quad (3.11)
\]

and (3.3) is satisfied. Then \( S \) and \( T \) have a common fixed point in \( X \).
Proof. By a method similar to that in the proof of Theorem 3.1, the result follows.

Theorem 3.3. Let $X$ be a complete metric space with a metric $d$ and let $p$ be a $w$-distance on $X$. Suppose that $T$ is a mapping of $X$ into $CB_p(X)$ and $\psi : X \to [0, \infty)$ is a mapping such that

$$p(u_1, u_2) \leq q \cdot \psi(x)$$

(3.12)

for all $x \in X$, $u_1 \in Tx$, $u_2 \in T^2x$ and some $q \in [0,1)$,

$$\sup \left\{ \frac{\psi(x)}{p(x, Tx)} : x \in X \right\} < \frac{1}{q},$$

(3.13)

$$\inf \{ p(x, u) + p(x, Tx) : x \in X \} > 0,$$

for every $u \in X$ with $u \notin Tu$. Then $T$ has a fixed point in $X$.

Proof. By a method similar to that in the proof of Theorem 3.1, the result follows.

Theorem 3.4. Let $X$ be a complete metric space with a metric $d$ and let $p$ be a $w$-distance on $X$. Suppose that $S$ and $T$ are self-mapping of $X$ and $\varphi : X \times X \to [0, \infty)$ is a mapping such that

$$\max \{ p(Sx, S^2x), p(Ty, T^2y) \} \leq q \cdot \varphi(x, y)$$

(3.14)

for all $x, y \in X$ and some $q \in [0,1)$,

$$\sup \left\{ \frac{\varphi(x, y)}{\min\left\{ p(x, Sx), p(y, Ty)\right\}} : x, y \in X \right\} < \frac{1}{q},$$

(3.15)

$$\inf \{ p(y, u) + p(x, Sx) + p(y, Ty) : x, y \in X \} > 0,$$

for every $u \in X$ with $u \neq Su$ or $u \neq Tu$. Then $S$ and $T$ have a common fixed point in $X$.

Proof. By a method similar to that in the proof of Theorem 3.1, the result follows.

From Theorem 3.1, we have the following corollary.

Corollary 3.5. Let $X$ be a complete metric space with a metric $d$ and let $p$ be a $w$-distance on $X$. Suppose that $S$ and $T$ are two mappings of $X$ into $CB_p(X)$ and $\varphi : X \times X \to [0, \infty)$ is a mapping such that

$$\max \\left\{ \sup\left\{ p(u_1, u_2) : u_1 \in Sx, u_2 \in S^2x \right\}, \sup\left\{ p(v_1, v_2) : v_1 \in Tx, v_2 \in T^2x \right\} \right\} \leq q \cdot \varphi(x, y)$$

(3.16)

for all $x, y \in X$ and some $q \in [0,1)$, and that (3.3) and (3.11) are satisfied. Then $S$ and $T$ have a common fixed point in $X$. 
From Theorem 3.3, we have the following corollaries.

**Corollary 3.6.** Let $X$ be a complete metric space with a metric $d$ and let $p$ be a $w$-distance on $X$. Suppose that $T$ is a mapping of $X$ into $CB_p(X)$ and $\psi : X \to [0, \infty)$ is a mapping such that

$$\sup \{ p(u_1, u_2) : u_1 \in Tx, u_2 \in T^2x \} \leq q \cdot \psi(x)$$

(3.17)

for all $x \in X$ and some $q \in [0, 1)$, and that (3.13) is satisfied. Then $T$ has a fixed point in $X$.

**Corollary 3.7.** Let $X$ be a complete metric space with a metric $d$ and let $p$ be a $w$-distance on $X$. Suppose that $T$ is a self-mapping of $X$ and $\psi : X \to [0, \infty)$ is a mapping such that

$$p(Tx, T^2x) \leq q \cdot \psi(x)$$

(3.18)

for all $x \in X$ and some $q \in [0, 1)$,

$$\sup \left\{ \psi(x) \over p(x, Tx) : x \in X \right\} < {1 \over q},$$

$$\inf \{ p(x, u) + p(x, Tx) : x \in X \} > 0,$$

(3.19)

for every $u \in X$ with $u \neq Tu$. Then $T$ has a fixed point in $X$.

From Corollary 3.7, we have the following corollaries.

**Corollary 3.8** (see [3]). Let $X$ be a complete metric space with a metric $d$ and let $p$ be a $w$-distance on $X$. Suppose that $T$ is a self-mapping of $X$ such that

$$p(Tx, Ty) \leq q \cdot \max \{ p(x, y), p(x, Tx), p(y, Ty), p(x, Ty), p(y, Tx) \}$$

(3.20)

for all $x, y \in X$ and some $q \in [0, 1)$, and that

$$\inf \{ p(x, u) + p(x, Tx) : x \in X \} > 0$$

(3.21)

for every $u \in X$ with $u \neq Tu$. Then $T$ has a unique fixed point in $X$.

**Proof.** By (3.20) and Lemma 2.4(3), we have

$$\sup \{ p(T^i x, T^j x) : i, j \in \mathbb{N} \cup \{0\} \} < \infty$$

(3.22)

for every $x \in X$. Thus we may define a function $r : X \times X \to [0, \infty)$ by

$$r(x, y) = \max \{ p(T^i x, T^j x) : i, j \in \mathbb{N} \cup \{0\}, p(x, y) \}$$

(3.23)

for every $x, y \in X$. Clearly, $r$ is a $w$-distance on $X$. Let $x$ be a given element of $X$, then, by using Lemma 2.4(1), (3.20), and (3.23), we have

$$r(Tx, T^2x) = \sup \{ p(T^i x, T^j x) : i, j \in \mathbb{N} \} \leq q \cdot \sup \{ p(T^i x, T^j x) : i, j \in \mathbb{N} \cup \{0\} \}$$

(3.24)

and

$$= q \cdot r(x, Tx).$$
By (3.21) and (3.23), we obtain
\[
\inf \{ r(x,u) + r(x,Tx) : x \in X \} > 0 \tag{3.25}
\]
for every \( u \in X \) with \( u \neq Tu \). From (3.24), (3.25), and Corollary 3.7, \( T \) has a fixed point in \( X \). By (3.20) and Lemma 2.4, it is clear that the fixed point of \( T \) is unique. \( \square \)

**Corollary 3.9** (see [2]). Let \( X \) be a complete metric space, let \( p \) be a \( w \)-distance on \( X \), and let \( T \) be a mapping from \( X \) into itself. Suppose that there exists \( q \in [0,1) \) such that
\[
p(Tx, T^2x) \leq q \cdot p(x,Tx) \tag{3.26}
\]
for every \( x \in X \) and that
\[
\inf \{ p(x,y) + p(x,Tx) : x \in X \} > 0 \tag{3.27}
\]
for every \( y \in X \) with \( y \neq Ty \). Then \( T \) has a fixed point in \( X \).

**Proof.** Define \( \psi : X \to [0,\infty) \) by
\[
\psi(x) = p(x,Tx) \tag{3.28}
\]
for all \( x \in X \). Thus the conditions of Corollary 3.7 are satisfied. Hence \( T \) has a fixed point in \( X \). \( \square \)

From Corollary 3.8, we have the following corollary.

**Corollary 3.10** (see [1]). Let \( X \) be a complete metric space with a metric \( d \) and let \( T \) be a mapping from \( X \) into itself. Suppose that \( T \) is a quasicontraction, that is, there exists \( q \in [0,1) \) such that
\[
d(Tx, Ty) \leq q \cdot \max \{ d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx) \} \tag{3.29}
\]
for every \( x,y \in X \). Then \( T \) has a unique fixed point in \( X \).

**Proof.** It is clear that the metric \( d \) is a \( w \)-distance and
\[
\inf \{ d(x,y) + d(x,Tx) : x \in X \} > 0 \tag{3.30}
\]
for every \( y \in X \) with \( y \neq Ty \). Thus, by Corollary 3.8 or 3.9, \( T \) has a unique fixed point in \( X \). \( \square \)

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Space dynamics is a very general title that can accommodate a long list of activities. This kind of research started with the study of the motion of the stars and the planets back to the origin of astronomy, and nowadays it has a large list of topics. It is possible to make a division in two main categories: astronomy and astrodynamics. By astronomy, we can relate topics that deal with the motion of the planets, natural satellites, comets, and so forth. Many important topics of research nowadays are related to those subjects. By astrodynamics, we mean topics related to spaceflight dynamics.

It means topics where a satellite, a rocket, or any kind of man-made object is travelling in space governed by the gravitational forces of celestial bodies and/or forces generated by propulsion systems that are available in those objects. Many topics are related to orbit determination, propagation, and orbital maneuvers related to those spacecrafts. Several other topics that are related to this subject are numerical methods, nonlinear dynamics, chaos, and control.

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