We define Riemann-Liouville transform \( R_\alpha \) and its dual \( t^{R_\alpha} \) associated with two singular partial differential operators. We establish some results of harmonic analysis for the Fourier transform connected with \( R_\alpha \). Next, we prove inversion formulas for the operators \( R_\alpha, t^{R_\alpha} \) and a Plancherel theorem for \( t^{R_\alpha} \).

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1. Introduction

The mean operator is defined for a continuous function \( f \) on \( \mathbb{R}^2 \), even with respect to the first variable by

\[
R_0(f)(r,x) = \frac{1}{2\pi} \int_0^{2\pi} f(r \sin \theta, x + r \cos \theta) d\theta,
\]

which means that \( R_0(f)(r,x) \) is the mean value of \( f \) on the circle centered at \((0,x)\) and radius \( r \). The dual of the mean operator \( t^{R_0} \) is defined by

\[
t^{R_0}(f)(r,x) = \frac{1}{\pi} \int_\mathbb{R} f(\sqrt{r^2 + (x-y)^2}, y) dy.
\]

The mean operator \( R_0 \) and its dual \( t^{R_0} \) play an important role and have many applications, for example, in image processing of the so-called synthetic aperture radar (SAR) data [11, 12] or in the linearized inverse scattering problem in acoustics [6].

Our purpose in this work is to define and study integral transforms which generalize the operators \( R_0 \) and \( t^{R_0} \). More precisely, we consider the following singular partial differential operators:

\[
\Delta_1 = \frac{\partial}{\partial x},
\]

\[
\Delta_2 = \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r} - \frac{\partial^2}{\partial x^2}, \quad (r,x) \in ]0, +\infty[ \times \mathbb{R}, \alpha \geq 0.
\]
Inversion formulas for Riemann-Liouville transform

We associate to $\Delta_1$ and $\Delta_2$ the Riemann-Liouville transform $\mathcal{R}_\alpha$, defined on $C_\ast(\mathbb{R}^2)$ (the space of continuous functions on $\mathbb{R}^2$, even with respect to the first variable) by

$$
\mathcal{R}_\alpha(f)(r,x) = \begin{cases}
\frac{\alpha}{\pi} \int_{-1}^{1} f(rs\sqrt{1-t^2},x+rt) 	imes (1-t^2)^{\alpha-1/2}(1-s^2)^{\alpha-1/2} \, dt \, ds, & \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{-1}^{1} f(r\sqrt{1-t^2},x+rt) \frac{dt}{\sqrt{1-t^2}}, & \text{if } \alpha = 0.
\end{cases}
$$

(1.4)

The dual operator $i\mathcal{R}_\alpha$ is defined on the space $\mathcal{L}_\ast(\mathbb{R}^2)$ (the space of infinitely differentiable functions on $\mathbb{R}^2$, rapidly decreasing together with all their derivatives, even with respect to the first variable) by

$$
i\mathcal{R}_\alpha(f)(r,x) = \begin{cases}
\frac{2\alpha}{\pi} \int_{r}^{\infty} \int_{-\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} f(u,x+v)(u^2-v^2-r^2)^{\alpha-1} \, ududv, & \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{\mathbb{R}} f(\sqrt{r^2+(x-y)^2},y) \, dy, & \text{if } \alpha = 0.
\end{cases}
$$

(1.5)

For more general fractional integrals and fractional differential equations, we can see the works of Debnath [3, 4] and Debnath with Bhatta [5].

We establish for the operators $\mathcal{R}_\alpha$ and $i\mathcal{R}_\alpha$ the same results given by Helgason, Ludwig, and Solmon for the classical Radon transform on $\mathbb{R}^2$ [10, 14, 17] and we find the results given in [15] for the spherical mean operator. Especially

(i) we give some harmonic analysis results related to the Fourier transform associated with the Riemann-Liouville transform $\mathcal{R}_\alpha$;

(ii) we define and characterize some spaces of the functions on which $\mathcal{R}_\alpha$ and $i\mathcal{R}_\alpha$ are isomorphisms;

(iii) we give the following inversion formulas for $\mathcal{R}_\alpha$ and $i\mathcal{R}_\alpha$:

$$
f = \mathcal{R}_\alpha K_1^1 \mathcal{R}_\alpha(f), \quad f = K_1^1 \mathcal{R}_\alpha \mathcal{R}_\alpha(f),
$$

$$
f = i\mathcal{R}_\alpha K_2^2 \mathcal{R}_\alpha(f), \quad f = K_2^2 \mathcal{R}_\alpha i\mathcal{R}_\alpha(f),
$$

(1.6)

where $K_1^1$ and $K_2^2$ are integro-differential operators;

(iv) we establish a Plancherel theorem for $i\mathcal{R}_\alpha$;

(v) we show that $\mathcal{R}_\alpha$ and $i\mathcal{R}_\alpha$ are transmutation operators.

This paper is organized as follows. In Section 2, we show that for $(\mu, \lambda) \in \mathbb{C}^2$, the differential system

$$
\Delta_1 u(r,x) = -i\lambda u(r,x),
$$

$$
\Delta_2 u(r,x) = -\mu^2 u(r,x),
$$

(1.7)

$$
u(0,0) = 1, \quad \frac{\partial u}{\partial r}(0,x) = 0, \quad \forall x \in \mathbb{R},$$

admits a unique solution \( \varphi_{\mu, \lambda} \) given by

\[
\varphi_{\mu, \lambda}(r, x) = j_{\alpha}(r \sqrt{\mu^2 + \lambda^2}) \exp(-i \lambda x),
\]

where \( j_{\alpha} \) is the modified Bessel function defined by

\[
j_{\alpha}(s) = 2^{\alpha} \Gamma(\alpha + 1) \frac{J_{\alpha}(s)}{s^\alpha},
\]

and \( f_{\alpha} \) is the Bessel function of first kind and index \( \alpha \). Next, we prove a Mehler integral representation of \( \varphi_{\mu, \lambda} \) and give some properties of \( R_{\alpha} \).

In Section 3, we define the Fourier transform \( \mathcal{F}_{\alpha} \) connected with \( R_{\alpha} \), and we establish some harmonic analysis results (inversion formula, Plancherel theorem, Paley-Wiener theorem) which lead to new properties of the operator \( R_{\alpha} \) and its dual \( {'}R_{\alpha} \).

In Section 4, we characterize some subspaces of \( \mathcal{S}_+(\mathbb{R}^2) \) on which \( R_{\alpha} \) and \( {'}R_{\alpha} \) are isomorphisms, and we prove the inversion formulas cited below where the operators \( K_{\alpha}^1 \) and \( K_{\alpha}^2 \) are given in terms of Fourier transforms. Next, we introduce fractional powers of the Bessel operator,

\[
\ell_{\alpha} = \frac{\partial^2}{\partial r^2} + \frac{2\alpha + 1}{r} \frac{\partial}{\partial r},
\]

and the Laplacian operator,

\[
\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x^2},
\]

that we use to simplify \( K_{\alpha}^1 \) and \( K_{\alpha}^2 \).

Finally, we prove the following Plancherel theorem for \( {'}R_{\alpha} \):

\[
\int_\mathbb{R} \int_0^{+\infty} |f(r, x)|^2 r^{2\alpha + 1} dr dx = \int_\mathbb{R} \int_0^{+\infty} |K_{\alpha}^3({'}R_{\alpha}(f))(r, x)|^2 dr dx,
\]

where \( K_{\alpha}^3 \) is an integro-differential operator.

In Section 5, we show that \( R_{\alpha} \) and \( {'}R_{\alpha} \) satisfy the following relations of permutation:

\[
{'}R_{\alpha}(\Delta_2 f) = \frac{\partial^2}{\partial r^2} {'}R_{\alpha}(f), \quad {'}R_{\alpha}(\Delta_1 f) = \Delta_1 {'}R_{\alpha}(f),
\]

\[
\Delta_2 R_{\alpha}(f) = R_{\alpha}\left(\frac{\partial^2 f}{\partial r^2}\right), \quad \Delta_1 R_{\alpha}(f) = R_{\alpha}(\Delta_1 f).
\]

2. Riemann-Liouville transform and its dual associated with the operators \( \Delta_1 \) and \( \Delta_2 \)

In this section, we define the Riemann-Liouville transform \( R_{\alpha} \) and its dual \( {'}R_{\alpha} \), and we give some properties of these operators. It is well known [21] that for every \( \lambda \in \mathbb{C} \), the system

\[
\ell_{\alpha} \varphi(r) = -\lambda^2 \varphi(r);
\]

\[
\varphi(0) = 1; \quad \varphi'(0) = 0,
\]
where $\ell_\alpha$ is the Bessel operator, admits a unique solution, that is, the modified Bessel function $r \mapsto j_\alpha(r\lambda)$. Thus, for all $(\mu, \lambda) \in \mathbb{C} \times \mathbb{C}$, the system

$$
\begin{align*}
\Delta_1 u(r, x) &= -i\lambda u(r, x), \\
\Delta_2 u(r, x) &= -\mu^2 u(r, x),
\end{align*}
$$

admits the unique solution given by

$$q_{\mu, \lambda}(r, x) = j_\alpha(r\sqrt{\mu^2 + \lambda^2}) \exp(-i\lambda x).$$

(2.3)

The modified Bessel function $j_\alpha$ has the Mehler integral representation, (we refer to [13, 21])

$$j_\alpha(s) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_{-1}^{1} \left(1 - t^2\right)^{\alpha-1/2} \exp(-ist)dt. $$

(2.4)

In particular,

$$\forall k \in \mathbb{N}, \forall s \in \mathbb{R}, \quad \left| j^{(k)}_\alpha(s) \right| \leq 1. $$

(2.5)

On the other hand,

$$\sup_{r \in \mathbb{R}} | j_\alpha(r\lambda) | = 1 \quad \text{iff} \quad \lambda \in \mathbb{R}. $$

(2.6)

This involves that

$$\sup_{(r, x) \in \mathbb{R}^2} | q_{\mu, \lambda}(r, x) | = 1 \quad \text{iff} \quad (\mu, \lambda) \in \Gamma, $$

(2.7)

where $\Gamma$ is the set defined by

$$\Gamma = \mathbb{R}^2 \cup \{ (i\mu, \lambda); (\mu, \lambda) \in \mathbb{R}^2, |\mu| \leq |\lambda| \}. $$

(2.8)

**Proposition 2.1.** The eigenfunction $q_{\mu, \lambda}$ given by (2.3) has the following Mehler integral representation:

$$q_{\mu, \lambda}(r, x) = \begin{cases}
\frac{\alpha}{\pi} \int_{-1}^{1} \cos \left(\mu rs\sqrt{1 - t^2}\right) \exp \left(-i\lambda(x + rt)\right) \left(1 - t^2\right)^{\alpha-1/2} \left(1 - s^2\right)^{\alpha-1} dt ds, & \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_{-1}^{1} \cos \left(r\sqrt{1 - t^2}\right) \exp \left(-i\lambda(x + rt)\right) \frac{dt}{\sqrt{1 - t^2}}, & \text{if } \alpha = 0.
\end{cases} $$

(2.9)
Proof. From the following expansion of the function $j_\alpha$:

$$j_\alpha(s) = 2^\alpha \Gamma(\alpha + 1) \frac{J_\alpha(s)}{s^\alpha} = \Gamma(\alpha + 1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(\alpha + k + 1)} \left( \frac{s}{2} \right)^{2k}, \quad (2.10)$$

we deduce that

$$j_\alpha\left(r \sqrt{\mu^2 + \lambda^2}\right) = \Gamma(\alpha + 1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(k + \alpha + 1)} \left( \frac{r \mu}{2} \right)^{2k} j_{\alpha+k}(r \lambda), \quad (2.11)$$

and from the equality (2.4), we obtain

$$j_\alpha\left(r \sqrt{\mu^2 + \lambda^2}\right) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_{-1}^{1} j_{\alpha-1/2}\left(r \mu \sqrt{1-t^2}\right) \exp(-ir\lambda t)(1-t^2)^{\alpha-1/2} dt. \quad (2.12)$$

Then, the results follow by using again the relation (2.4) for $\alpha > 0$, and from the fact that

$$j_{-1/2}(s) = \cos s, \quad \text{for} \ \alpha = 0. \quad (2.13) \square$$

Definition 2.2. The Riemann-Liouville transform $\mathcal{R}_\alpha$ associated with the operators $\Delta_1$ and $\Delta_2$ is the mapping defined on $\mathcal{C}_\ast(\mathbb{R}^2)$ by the following. For all $(r, x) \in \mathbb{R}^2$,

$$\mathcal{R}_\alpha(f)(r, x) = \begin{cases} \frac{\alpha}{\pi} \int_{-1}^{1} f\left(rs \sqrt{1-t^2}, x + rt\right) \\ \times (1-t^2)^{\alpha-1/2}(1-s^2)^{\alpha-1} dt ds, & \text{if} \ \alpha > 0, \\ \frac{1}{\pi} \int_{-1}^{1} f\left(r \sqrt{1-t^2}, x + rt\right) \frac{dt}{\sqrt{1-t^2}}, & \text{if} \ \alpha = 0. \end{cases} \quad (2.14)$$

Remark 2.3. (i) From Proposition 2.1 and Definition 2.2, we have

$$\varphi_{\mu,\lambda}(r, x) = \mathcal{R}_\alpha\left(\cos(\mu \cdot) \exp(-i\lambda \cdot)\right)(r, x). \quad (2.15)$$

(ii) We can easily see, as in [2], that the transform $\mathcal{R}_\alpha$ is continuous and injective from $\mathcal{C}_\ast(\mathbb{R}^2)$ (the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable) into itself.
Inversion formulas for Riemann-Liouville transform

Lemma 2.4. For \( f \in \mathcal{C}_b(\mathbb{R}^2), \) \( f \) bounded, and \( g \in \mathcal{S}_*(\mathbb{R}^2), \)

\[
\int_\mathbb{R} \int_0^{+\infty} \mathcal{R}_\alpha(f)(r,x)g(r,x)r^{2\alpha+1} \, dr \, dx = \int_\mathbb{R} \int_0^{+\infty} f(r,x) \mathcal{R}_\alpha^*(g)(r,x) \, dr \, dx,
\]  

(2.16)

where \( \mathcal{R}_\alpha^* \) is the dual transform defined by

\[
\mathcal{R}_\alpha^*(g)(r,x) = \begin{cases} 
\frac{2\alpha}{\pi} \int_r^{+\infty} \int_{\sqrt{u^2-r^2}}^{\sqrt{u^2-r^2}} g(u,v)(u^2-v^2-r^2)^{\alpha-1} \, u \, du \, dv, & \text{if } \alpha > 0, \\
\frac{1}{\pi} \int_\mathbb{R} g\left(\sqrt{r^2+(x-y)^2}, y\right) \, dy, & \text{if } \alpha = 0.
\end{cases}
\]

(2.17)

To obtain this lemma, we use Fubini’s theorem and an adequate change of variables.

Remark 2.5. By a simple change of variables, we have

\[
\mathcal{R}_0(f)(r,x) = \frac{1}{2\pi} \int_0^{2\pi} f(r \sin \theta, x + r \cos \theta) \, d\theta.
\]

(2.18)

3. Fourier transform associated with Riemann-Liouville operator

In this section, we define the Fourier transform associated with the operator \( \mathcal{R}_\alpha, \) and we give some results of harmonic analysis that we use in the next sections.

We denote by

(i) \( d\nu(r,x) \) the measure defined on \([0, +\infty[ \times \mathbb{R}\) by

\[
d\nu(r,x) = \frac{1}{\sqrt{2\pi} 2^a \Gamma(\alpha + 1)} r^{2\alpha+1} \, dr \otimes dx,
\]

(3.1)

(ii) \( L^1(d\nu) \) the space of measurable functions \( f \) on \([0, +\infty[ \times \mathbb{R}\) satisfying

\[
\|f\|_{1,\nu} = \int_\mathbb{R} \int_0^{+\infty} |f(r,x)| \, d\nu(r,x) < +\infty.
\]

(3.2)

Definition 3.1. (i) The translation operator associated with Riemann-Liouville transform is defined on \( L^1(d\nu) \) by the following. For all \( (r,x), (s,y) \in [0, +\infty[ \times \mathbb{R}, \)

\[
\mathcal{T}(r,x)f(s,y) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_0^\pi f\left(\sqrt{r^2+s^2+2rs \cos \theta}, x+y\right) \sin^{2\alpha} \theta \, d\theta.
\]

(3.3)

(ii) The convolution product associated with the Riemann-Liouville transform of \( f, g \in L^1(d\nu) \) is defined by the following. For all \( (r,x) \in [0, +\infty[ \times \mathbb{R}, \)

\[
f \ast g(r,x) = \int_\mathbb{R} \int_0^{+\infty} \mathcal{T}(r,x)f(s,y)g(s,y) \, d\nu(s,y),
\]

(3.4)

where \( \hat{f}(s,y) = f(s,-y). \)
We have the following properties.

(i) Since
\[ \forall r, s \geq 0, \quad j_\alpha(\sqrt{r^2 + s^2 + 2rs \cos \theta}) = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi} \Gamma(\alpha + 1/2)} \int_0^\pi j_\alpha(\sqrt{r^2 + s^2 + 2rs \cos \theta} \sin^2 \alpha \theta) d\theta, \] (3.5)
we refer to \[21\]) we deduce that the eigenfunction \( \varphi_{\mu, \lambda} \) defined by the relation (2.3) satisfies the product formula
\[ \mathcal{T}_{(r, x)} \varphi_{\mu, \lambda}(s, y) = \varphi_{\mu, \lambda}(r, x) \varphi_{\mu, \lambda}(s, y). \] (3.6)

(ii) If \( f \in L^1(d\nu) \), then for all \((r, x) \in [0, +\infty[ \times \mathbb{R}, \mathcal{T}_{(r, x)} f \) belongs to \( L^1(d\nu) \), and we have
\[ \| \mathcal{T}_{(r, x)} f \|_{1, \nu} \leq \| f \|_{1, \nu}. \] (3.7)

(iii) For \( f, g \in L^1(d\nu) \), \( f \# g \) belongs to \( L^1(d\nu) \), and the convolution product is commutative and associative.

(iv) For \( f, g \in L^1(d\nu) \),
\[ \| f \# g \|_{1, \nu} \leq \| f \|_{1, \nu} \| g \|_{1, \nu}. \] (3.8)

**Definition 3.2.** The Fourier transform associated with the Riemann-Liouville operator is defined by
\[ \forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(f)(\mu, \lambda) = \int_\mathbb{R} \int_0^{+\infty} f(r, x) \varphi_{\mu, \lambda}(r, x) d\nu(r, x), \] (3.9)
where \( \Gamma \) is the set defined by the relation (2.8).

We have the following properties.

(i) Let \( f \) be in \( L^1(d\nu) \). For all \((r, x) \in [0, +\infty[ \times \mathbb{R}, \) we have
\[ \forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(\mathcal{T}_{(r, -x)} f)(\mu, \lambda) = \varphi_{\mu, \lambda}(r, x) \mathcal{F}_\alpha(f)(\mu, \lambda). \] (3.10)

(ii) For \( f, g \in L^1(d\nu) \), we have
\[ \forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(f \# g)(\mu, \lambda) = \mathcal{F}_\alpha(f)(\mu, \lambda) \mathcal{F}_\alpha(g)(\mu, \lambda). \] (3.11)

(iii) For \( f \in L^1(d\nu) \), we have
\[ \forall (\mu, \lambda) \in \Gamma, \quad \mathcal{F}_\alpha(f)(\mu, \lambda) = B \circ \tilde{\mathcal{F}}_\alpha(f)(\mu, \lambda), \] (3.12)
where
\[ \forall (\mu, \lambda) \in \mathbb{R}^2, \quad \tilde{\mathcal{F}}_\alpha(f)(\mu, \lambda) = \int_\mathbb{R} \int_0^{+\infty} f(r, x) j_\alpha(r \mu) \exp(-i \lambda x) d\nu(r, x), \] (3.13)
\[ \forall (\mu, \lambda) \in \Gamma, \quad Bf(\mu, \lambda) = f(\sqrt{\mu^2 + \lambda^2}, \lambda). \]
Inversion formulas for Riemann-Liouville transform

3.1. Inversion formula and Plancherel theorem for $\mathfrak{F}_\alpha$. We denote by (see [15])

(i) $\mathcal{S}_\alpha(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$ rapidly decreasing together with all their derivatives, even with respect to the first variable;

(ii) $\mathcal{S}_\alpha(\Gamma)$ the space of functions $f : \Gamma \to \mathbb{C}$ infinitely differentiable, even with respect to the first variable and rapidly decreasing together with all their derivatives, that is, for all $k_1, k_2, k_3 \in \mathbb{N}$,

$$\sup_{(\mu, \lambda) \in \Gamma} (1 + |\mu|^2 + |\lambda|^2)^{k_1} \left| \left( \frac{\partial}{\partial \mu} \right)^{k_2} \left( \frac{\partial}{\partial \lambda} \right)^{k_3} f(\mu, \lambda) \right| < +\infty,$$

where

$$\frac{\partial f}{\partial \mu}(\mu, \lambda) = \begin{cases} \frac{\partial}{\partial r}(f(r, \lambda)), & \text{if } \mu = r \in \mathbb{R}, \\ \frac{1}{i} \frac{\partial}{\partial t}(f(it, \lambda)), & \text{if } \mu = it, |t| \leq |\lambda|. \end{cases}$$

Each of these spaces is equipped with its usual topology:

(i) $L^2(d\nu)$ the space of measurable functions on $[0, +\infty) \times \mathbb{R}$ such that

$$\|f\|_{2,\nu} = \left( \int_\mathbb{R} \int_0^{+\infty} |f(r, x)|^2 d\nu(r, x) \right)^{1/2} < +\infty;$$

(ii) $d\gamma(\mu, \lambda)$ the measure defined on $\Gamma$ by

$$\int_\Gamma f(\mu, \lambda) d\gamma(\mu, \lambda)$$

$$= \frac{1}{\sqrt{2\pi 2^\alpha \Gamma(\alpha + 1)}} \left\{ \int_\mathbb{R} \int_0^{+\infty} f(\mu, \lambda)(\mu^2 + \lambda^2)^\alpha \mu \, d\mu \, d\lambda + \int_\mathbb{R} \int_0^{+\infty} f(i\mu, \lambda)(\lambda^2 - \mu^2)^\alpha \mu \, d\mu \, d\lambda \right\};$$

(iii) $L^p(d\gamma)$, $p = 1, p = 2$, the space of measurable functions on $\Gamma$ satisfying

$$\|f\|_{p,\gamma} = \left( \int_\Gamma \|f(\mu, \lambda)\|^p d\gamma(\mu, \lambda) \right)^{1/p} < +\infty.$$

Remark 3.3. It is clear that a function $f$ belongs to $L^1(d\nu)$ if, and only if, the function $Bf$ belongs to $L^1(d\gamma)$, and we have

$$\int_\Gamma Bf(\mu, \lambda) d\gamma(\mu, \lambda) = \int_\mathbb{R} \int_0^{+\infty} f(r, x) d\nu(r, x).$$

Proposition 3.4 (inversion formula for $\mathfrak{F}_\alpha$). Let $f \in L^1(d\nu)$ such that $\mathfrak{F}_\alpha(f)$ belongs to $L^1(d\gamma)$, then for almost every $(r, x) \in [0, +\infty) \times \mathbb{R}$,

$$f(r, x) = \int_\Gamma \mathfrak{F}_\alpha(f)(\mu, \lambda) \phi_{\mu, \lambda}(r, x) d\gamma(\mu, \lambda).$$
Proof. From [9, 19], one can see that if \( f \in L^1(d\nu) \) is such that \( \tilde{\mathcal{F}}_\alpha(f) \in L^1(d\nu) \), then for almost every \((r,x) \in [0, +\infty[ \times \mathbb{R}\),

\[
f(r,x) = \int_\mathbb{R} \int_0^{+\infty} \tilde{\mathcal{F}}_\alpha(f)(\mu,\lambda) j_\alpha(r\mu) \exp(i\lambda x) d\nu(\mu,\lambda).
\]

(3.21)

Then, the result follows from the relation (3.12) and Remark 3.3. \( \square \)

Theorem 3.5. (i) The Fourier transform \( \tilde{\mathcal{F}}_\alpha \) is an isomorphism from \( \mathcal{S}_*(\mathbb{R}^2) \) onto \( \mathcal{S}_*(\Gamma) \).

(ii) (Plancherel formula) for \( f \in \mathcal{S}_*(\mathbb{R}^2) \),

\[
\|\tilde{\mathcal{F}}_\alpha(f)\|_{2,\nu} = \|f\|_{2,\nu}.
\]

(3.22)

(iii) (Plancherel theorem) the transform \( \tilde{\mathcal{F}}_\alpha \) can be extended to an isometric isomorphism from \( L^2(d\nu) \) onto \( L^2(d\gamma) \).

Proof. This theorem follows from the relation (3.12), Remark 3.3, and the fact that \( \tilde{\mathcal{F}}_\alpha \) is an isomorphism from \( \mathcal{S}_*(\mathbb{R}^2) \) onto itself, satisfying that for all \( f \in \mathcal{S}_*(\mathbb{R}^2) \),

\[
\|\tilde{\mathcal{F}}_\alpha(f)\|_{2,\nu} = \|f\|_{2,\nu}.
\]

(3.23) \( \square \)

Lemma 3.6. For \( f \in \mathcal{S}_*(\mathbb{R}^2) \),

\[
\forall (\mu,\lambda) \in \mathbb{R}^2, \quad \tilde{\mathcal{F}}_\alpha(f)(\mu,\lambda) = \Lambda_\alpha \circ \mathcal{R}_\alpha(f)(\mu,\lambda),
\]

(3.24)

where \( \mathcal{R}_\alpha \) is the dual transform of the Riemann–Liouville operator, and \( \Lambda_\alpha \) is a constant multiple of the classical Fourier transform on \( \mathbb{R}^2 \) defined by

\[
\Lambda_\alpha(f)(\mu,\lambda) = \int_\mathbb{R} \int_0^{+\infty} f(r,x) \cos(r\mu) \exp(-i\lambda x) dm(r,x),
\]

(3.25)

where \( dm(r,x) \) is the measure defined on \([0, +\infty[ \times \mathbb{R}\) by

\[
dm(r,x) = \frac{1}{\sqrt{2\pi 2^\alpha \Gamma(\alpha + 1)}} dr \otimes dx.
\]

(3.26)

This lemma follows from the relation (2.15) and Lemma 2.4.

Using the relation (3.12) and the fact that the mapping \( B \) is continuous from \( \mathcal{S}_*(\mathbb{R}^2) \) into itself, we deduce that the Fourier transform \( \tilde{\mathcal{F}}_\alpha \) is continuous from \( \mathcal{S}_*(\mathbb{R}^2) \) into itself. On the other hand, \( \Lambda_\alpha \) is an isomorphism from \( \mathcal{S}_*(\mathbb{R}^2) \) onto itself. Then, Lemma 3.6 implies that the dual transform \( \mathcal{R}_\alpha \) maps continuously \( \mathcal{S}_*(\mathbb{R}^2) \) into itself.

Proposition 3.7. (i) \( \mathcal{R}_\alpha \) is not injective when applied to \( \mathcal{S}_*(\mathbb{R}^2) \).

(ii) \( \mathcal{R}_\alpha(\mathcal{S}_*(\mathbb{R}^2)) = \mathcal{S}_*(\mathbb{R}^2) \).
Proof. (i) Let $g \in \mathcal{F}_*(\mathbb{R}^2)$ such that $\text{supp} \, g \subset \{(r, x) \in \mathbb{R}^2, \ |r| \leq |x|\}, \ g \neq 0$.

Since $\mathcal{F}_a$ is an isomorphism from $\mathcal{F}_*(\mathbb{R}^2)$ onto itself, there exists $f \in \mathcal{F}_*(\mathbb{R}^2)$ such that $\mathcal{F}_a(f) = g$. From the relation (3.12) and Lemma 3.6, we deduce that $\mathcal{F}_*(f) = 0$.

(ii) We obtain the result by the same way as in [1]. □

3.2. Paley-Wiener theorem. We denote by

(i) $\mathcal{D}_*(\mathbb{R}^2)$ the space of infinitely differentiable functions on $\mathbb{R}^2$, even with respect to the first variable, and with compact support;

(ii) $\mathcal{H}_*(\mathbb{C}^2)$ the space of entire functions $f : \mathbb{C}^2 \to \mathbb{C}$, even with respect to the first variable rapidly decreasing of exponential type, that is, there exists a positive constant $M$, such that for all $k \in \mathbb{N}$,

$$\sup_{(\mu, \lambda) \in \mathbb{C}^2} (1 + |\mu|^2 + |\lambda|^2)^k |f(\mu, \lambda)| \exp \left( -M(|\text{Im}\mu| + |\text{Im}\lambda|) \right) < +\infty; \quad (3.27)$$

(iii) $\mathcal{H}_{*,0}(\mathbb{C}^2)$ the subspace of $\mathcal{H}_*(\mathbb{C}^2)$, consisting of functions $f : \mathbb{C}^2 \to \mathbb{C}$, such that for all $k \in \mathbb{N}$,

$$\sup_{(\mu, \lambda) \in \mathbb{R}^2, |\mu| \leq |\lambda|} (1 - \mu^2 + 2\lambda^2)^k |f(i\mu, \lambda)| < +\infty; \quad (3.28)$$

(iv) $\mathcal{E}_*(\mathbb{R}^2)$ the space of distributions on $\mathbb{R}^2$, even with respect to the first variable, and with compact support;

(v) $\mathcal{H}_*(\mathbb{C}^2)$ the space of entire functions $f : \mathbb{C}^2 \to \mathbb{C}$, even with respect to the first variable, slowly increasing of exponential type, that is, there exists a positive constant $M$ and an integer $k$, such that

$$\sup_{(\mu, \lambda) \in \mathbb{C}^2} (1 + |\mu|^2 + |\lambda|^2)^{-k} |f(\mu, \lambda)| \exp \left( -M(|\text{Im}\mu| + |\text{Im}\lambda|) \right) < +\infty; \quad (3.29)$$

(vi) $\mathcal{H}_{*,0}(\mathbb{C}^2)$ the subspace of $\mathcal{H}_*(\mathbb{C}^2)$, consisting of functions $f : \mathbb{C}^2 \to \mathbb{C}$, such that there exists an integer $k$, satisfying

$$\sup_{(\mu, \lambda) \in \mathbb{R}^2, |\mu| \leq |\lambda|} (1 - \mu^2 + 2\lambda^2)^{-k} |f(i\mu, \lambda)| < +\infty. \quad (3.30)$$

Each of these spaces is equipped with its usual topology.

Definition 3.8. The Fourier transform associated with the Riemann-Liouville operator is defined on $\mathcal{E}_*(\mathbb{R}^2)$ by

$$\forall (\mu, \lambda) \in \mathbb{C}^2, \ \mathcal{F}_a(T)(\mu, \lambda) = \langle T, \varphi_{\mu, \lambda} \rangle. \quad (3.31)$$
Proposition 3.9. For every $T \in \mathcal{E}'_a(\mathbb{R}^2)$,
\[ \forall (\mu, \lambda) \in \mathbb{C}^2, \quad \tilde{\mathcal{F}}_a(T)(\mu, \lambda) = B \circ \tilde{\mathcal{S}}_a(T)(\mu, \lambda), \tag{3.32} \]
where
\[ \forall (\mu, \lambda) \in \mathbb{C}^2, \quad \tilde{\mathcal{F}}_a(T)(\mu, \lambda) = \langle T, j_\alpha(\mu) \exp(-i\lambda) \rangle, \tag{3.33} \]
and $B$ is the transform defined by the relation (3.12).

Using [7, Lemma 2] (see also [15]) and the fact that $\tilde{\mathcal{F}}_a$ is an isomorphism from $\mathcal{D}_*(\mathbb{R}^2)$ (resp., $\mathcal{E}'_*(\mathbb{R}^2)$) onto $\mathcal{H}_*(\mathbb{C}^2)$ (resp., $\mathcal{H}_*(\mathbb{C}^2)$), we deduce the following theorem.

Theorem 3.10 (of Paley-Wiener). The Fourier transform $\tilde{\mathcal{F}}_a$ is an isomorphism
(i) from $\mathcal{D}_*(\mathbb{R}^2)$ onto $\mathcal{H}_*(\mathbb{C}^2)$;
(ii) from $\mathcal{E}'_*(\mathbb{R}^2)$ onto $\mathcal{H}_*(\mathbb{C}^2)$.

From Lemma 3.6, Theorem 3.10, and the fact that $\Lambda_\alpha$ is an isomorphism from $\mathcal{D}_*(\mathbb{R}^2)$ onto $\mathcal{H}_*(\mathbb{C}^2)$, we have the following corollary.

Corollary 3.11. (i) $t^\mathcal{R}_\alpha$ maps injectively $\mathcal{D}_*(\mathbb{R}^2)$ into itself.
(ii) $t^\mathcal{R}_\alpha(\mathcal{D}_*(\mathbb{R}^2)) \neq \mathcal{D}_*(\mathbb{R}^2)$.

4. Inversion formulas for $\mathcal{R}_a$ and $t^\mathcal{R}_\alpha$ and Plancherel theorem for $t^\mathcal{R}_\alpha$

In this section, we will define some subspaces of $\mathcal{S}_*(\mathbb{R}^2)$ on which $\mathcal{R}_a$ and $t^\mathcal{R}_\alpha$ are isomorphisms, and we give their inverse transforms in terms of integro-differential operators. Next, we establish Plancherel theorem for $t^\mathcal{R}_\alpha$.

We denote by
(i) $\mathcal{N}$ the subspace of $\mathcal{S}_*(\mathbb{R}^2)$, consisting of functions $f$ satisfying
\[ \forall k \in \mathbb{N}, \forall x \in \mathbb{R}, \quad \left( \frac{\partial}{\partial r^2} \right)^k f(0, x) = 0, \tag{4.1} \]
where
\[ \frac{\partial}{\partial r^2} = \frac{1}{r} \frac{\partial}{\partial r}; \tag{4.2} \]
(ii) $\mathcal{S}_{*,0}(\mathbb{R}^2)$ the subspace of $\mathcal{S}_*(\mathbb{R}^2)$, consisting of functions $f$, such that
\[ \forall k \in \mathbb{N}, \forall x \in \mathbb{R}, \quad \int_0^{+\infty} f(r, x) r^{2k} dr = 0; \tag{4.3} \]
(iii) $\mathcal{S}_{*,a}^0(\mathbb{R}^2)$ the subspace of $\mathcal{S}_*(\mathbb{R}^2)$, consisting of functions $f$, such that
\[ \text{supp} \tilde{\mathcal{F}}_a(f) \subset \{ (\mu, \lambda) \in \mathbb{R}^2; |\mu| \geq |\lambda| \}. \tag{4.4} \]
Lemma 4.1. (i) The mapping $\Lambda_\alpha$ is an isomorphism from $\mathcal{F}_{*,0}(\mathbb{R}^2)$ onto $\mathcal{N}$.
(ii) The subspace $\mathcal{N}$ can be written as
\[\mathcal{N} = \left\{ f \in \mathcal{F}_{*,0}(\mathbb{R}^2); \forall k \in \mathbb{N}, \forall x \in \mathbb{R}; \left( \frac{\partial}{\partial r} \right)^{2k} f(0,x) = 0 \right\}. \tag{4.5}\]

Proof. Let $f \in \mathcal{F}_{*,0}(\mathbb{R}^2)$.
(i) For $\nu > -1$, we have
\[\left( \frac{\partial}{\partial \mu^2} \right)^k (j_\nu(r\mu)) = \frac{\Gamma(\nu + 1)}{2^k \Gamma(\nu + k + 1)} (-r^2)^k j_{\nu+k}(r\mu), \tag{4.6}\]
thus, from the expression of $\Lambda_\alpha$, given in Lemma 3.6, and the fact that $j_{-1/2}(s) = \cos s$, we obtain
\[\left( \frac{\partial}{\partial \mu^2} \right)^k (\Lambda_\alpha(f))(0,\lambda) = \frac{\sqrt{\pi}}{2^k \Gamma(k + 1/2)} (-1)^k \int_0^\infty \int_0^{+\infty} f(r,x) r^{2k} \exp(-i\lambda x) dm(r,x), \tag{4.7}\]
which gives the result.
(ii) The proof of (ii) is immediate. \hfill \Box

Theorem 4.2. (i) For all real numbers $\gamma$, the mappings
(i) $f \mapsto (r^2 + x^2)^\gamma f$
(ii) $f \mapsto |r|^\gamma f$
are isomorphisms from $\mathcal{N}$ onto itself.
(ii) For $f \in \mathcal{N}$, the function $g$ defined by
\[g(r,x) = \begin{cases} f\left(\sqrt{r^2 - x^2},x\right) & \text{if } |r| \geq |x|, \\ 0 & \text{otherwise}, \end{cases} \tag{4.8}\]
belongs to $\mathcal{F}_{*,0}(\mathbb{R}^2)$.

Proof. (i) Let $f \in \mathcal{N}$, by Leibnitz formula, we have
\[\left( \frac{\partial}{\partial r} \right)^{k_1} \left( \frac{\partial}{\partial x} \right)^{k_2} [ (r^2 + x^2)^\gamma f ](r,x) = \sum_{j=0}^{k_1} \sum_{i=0}^{k_2} C_{k_1}^j C_{k_2}^i P_j(r) P_i(x) (r^2 + x^2)^{\gamma - i - j} \frac{\partial^{k_1+k_2-i-j}}{\partial r^{k_1-j} \partial x^{k_2-j}} f(r,x), \tag{4.9}\]
where $P_i$ and $P_j$ are real polynomials.
Let \( n \in \mathbb{N} \) such that \( \gamma - k_1 - k_2 + n > 0 \). By Taylor formula and the fact that \( f \in \mathcal{N} \), we have

\[
\left( \frac{\partial}{\partial r} \right)^{k_1-j} \left( \frac{\partial}{\partial x} \right)^{k_2-i} (f)(r,x) = \frac{r^{2n}}{(2n-1)!} \int_0^1 (1-t)^{2n-1} \left( \frac{\partial}{\partial r} \right)^{k_1-j+2n} \left( \frac{\partial}{\partial x} \right)^{k_2-i} (f)(rt,x) \, dt
\]

(4.10)

\[
= -\frac{r^{2n}}{(2n-1)!} \int_1^{+\infty} (1-t)^{2n-1} \left( \frac{\partial}{\partial r} \right)^{k_1-j+2n} \left( \frac{\partial}{\partial x} \right)^{k_2-i} (f)(rt,x) \, dt,
\]

\[
\left( \frac{\partial}{\partial x} \right)^{k_2-i} \left( \frac{\partial}{\partial r} \right)^{k_1-j} f(r,x) = \frac{i^{2n}}{(2n-1)!} \int_0^1 (1-t)^{2n-1} \left( \frac{\partial}{\partial x} \right)^{k_2-i} \left( \frac{\partial}{\partial r} \right)^{k_1-j+2n} f(rt,x) \, dt
\]

(4.11)

The relations (4.9) and (4.11) imply that the function

\[
(r,x) \mapsto (r^2 + x^2)^\gamma f(r,x)
\]

(4.12)

belongs to \( \mathcal{N} \) and that the mapping

\[
f \mapsto (r^2 + x^2)^\gamma f \]

(4.13)

is continuous from \( \mathcal{N} \) onto itself. The inverse mapping is given by

\[
f \mapsto (r^2 + x^2)^{-\gamma} f.
\]

(4.14)

By the same way, we show that the mapping

\[
f \mapsto |r|^\gamma f
\]

(4.15)

is an isomorphism from \( \mathcal{N} \) onto itself.

(ii) Let \( f \in \mathcal{N} \), and

\[
g(r,x) = \begin{cases} f(\sqrt{r^2-x^2},x) & \text{if } |r| \geq |x|, \\ 0 & \text{if } |r| \leq |x|, \end{cases}
\]

(4.16)

we have

\[
\left( \frac{\partial}{\partial x} \right)^{k_2} \left( \frac{\partial}{\partial r} \right)^{k_1} (g)(r,x) = \sum_{j=0}^{k_1} P_j(r) \left( \sum_{p=0}^{k_1} Q_{p,q}(x) \left( \frac{\partial}{\partial x} \right)^p \left( \frac{\partial}{\partial r} \right)^{q+j} (f)(\sqrt{r^2-x^2},x) \right),
\]

(4.17)

where \( P_j \) and \( Q_{p,q} \) are real polynomials. This equality, together with the fact that \( f \) belongs to \( \mathcal{N} \), implies that \( g \) belongs to \( \mathcal{S}_* (\mathbb{R}^2) \). \( \square \)
Theorem 4.3. The Fourier transform $\mathcal{F}_\alpha$ associated with Riemann-Liouville transform is an isomorphism from $\mathcal{S}^0_*(\mathbb{R}^2)$ onto $\mathcal{N}$.

Proof. Let $f \in \mathcal{S}^0_*(\mathbb{R}^2)$. From the relation (3.12), we get

$$\left(\frac{\partial}{\partial \mu^2}\right)^k \tilde{\mathcal{F}}_\alpha f(0,\lambda) = B\left(\left(\frac{\partial}{\partial \mu^2}\right)^k \tilde{\mathcal{F}}_\alpha f\right)(0,\lambda)$$

$$= \left(\frac{\partial}{\partial \mu^2}\right)^k \tilde{\mathcal{F}}_\alpha f(\lambda,\lambda) = 0,$$

because $\text{supp} \tilde{\mathcal{F}}_\alpha f \subset \{ (\mu,\lambda) \in \mathbb{R}^2, |\mu| \geq |\lambda| \}$, this shows that $\mathcal{F}_\alpha$ maps injectively $\mathcal{S}^0_*(\mathbb{R}^2)$ into $\mathcal{N}$. On the other hand, let $h \in \mathcal{N}$ and

$$g(r,x) = \begin{cases} h(\sqrt{r^2-x^2},x) & \text{if } |r| \geq |x|, \\ 0 & \text{if } |r| \leq |x|. \end{cases} (4.19)$$

From Theorem 4.2(ii), $g$ belongs to $\mathcal{S}_*(\mathbb{R}^2)$, so there exists $f \in \mathcal{S}^0_*(\mathbb{R}^2)$ satisfying $\tilde{\mathcal{F}}_\alpha f = g$. Consequently, $f \in \mathcal{S}^0_*(\mathbb{R}^2)$ and $\mathcal{F}_\alpha f = h$. □

From Lemmas 3.6, 4.1, and Theorem 4.3, we deduce the following result.

Corollary 4.4. The dual transform $\mathcal{T}_\alpha$ is an isomorphism from $\mathcal{S}^0_*(\mathbb{R}^2)$ onto $\mathcal{S}_{*,0}^0(\mathbb{R}^2)$.

4.1. Inversion formula for $\mathcal{R}_\alpha$ and $\mathcal{T}_\alpha$

Theorem 4.5. (i) The operator $K^1_\alpha$ defined by

$$K^1_\alpha f(r,x) = \Lambda^{-1}_\alpha \left(\frac{\pi}{2^{2\alpha+1} \Gamma^2(\alpha+1)} (\mu^2 + \lambda^2)^\alpha |\mu| \Lambda_\alpha f\right)(r,x)$$

is an isomorphism from $\mathcal{S}_*(\mathbb{R}^2)$ onto itself.

(ii) The operator $K^2_\alpha$ defined by

$$K^2_\alpha g(r,x) = \tilde{\mathcal{F}}^{-1}_\alpha \left(\frac{\pi}{2^{2\alpha+1} \Gamma^2(\alpha+1)} (\mu^2 + \lambda^2)^\alpha |\mu| \tilde{\mathcal{F}}_\alpha g\right)(r,x)$$

is an isomorphism from $\mathcal{S}^0_*(\mathbb{R}^2)$ onto itself.

This theorem follows from Lemma 4.1, Theorems 4.2 and 4.3.

Theorem 4.6. (i) For $f \in \mathcal{S}_*(\mathbb{R}^2)$ and $g \in \mathcal{S}^0_*(\mathbb{R}^2)$, there exists the inversion formula for $\mathcal{R}_\alpha$:

$$g = \mathcal{R}_\alpha K^1_\alpha \mathcal{T}_\alpha f, \quad f = K^1_\alpha \mathcal{T}_\alpha \mathcal{R}_\alpha f.$$ (4.22)

(ii) For $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$ and $g \in \mathcal{S}^0_*(\mathbb{R}^2)$, there exists the inversion formula for $\mathcal{T}_\alpha$:

$$f = \mathcal{T}_\alpha K^2_\alpha \mathcal{R}_\alpha g, \quad g = K^2_\alpha \mathcal{R}_\alpha \mathcal{T}_\alpha f.$$ (4.23)
Proof. (i) Let \( g \in \mathcal{S}_\mathbb{R}^0(\mathbb{R}^2) \). From the relation (2.15), Proposition 3.4, Lemma 3.6, and Theorem 4.3, we have

\[
g(r, x) = \int_{\mathbb{R}} \int_0^{+\infty} (\mu^2 + \lambda^2)^{a/2} \mu \Lambda_\alpha \circ \mathcal{F}_\alpha (g)(\mu, \lambda) \cos(\mu \cdot r) \exp(i \lambda \cdot x) dm(\mu, \lambda)
\]

\[
= \mathcal{R}_\alpha \left( \int_{\mathbb{R}} \int_0^{+\infty} (\mu^2 + \lambda^2)^{a/2} \mu \Lambda_\alpha \circ \mathcal{F}_\alpha (g)(\mu, \lambda) \cos(\mu \cdot r) \exp(i \lambda \cdot x) dm(\mu, \lambda) \right)(r, x)
\]

\[
= \mathcal{R}_\alpha \mathcal{L}_\alpha^{-1} \left( \frac{\pi}{2^{2a+1} \Gamma^2(\alpha + 1)} (\mu^2 + \lambda^2)^{a/2} |\mu| \Lambda_\alpha \circ \mathcal{F}_\alpha (g) \right)(r, x)
\]

\[
= \mathcal{R}_\alpha K_\alpha^1 \mathcal{L}_\alpha(g)(r, x).
\]

(4.24)

This relation, together with Corollary 4.4 and Theorem 4.5(i), implies that \( \mathcal{R}_\alpha \) is an isomorphism from \( \mathcal{S}_\mathbb{R}^0(\mathbb{R}) \) onto \( \mathcal{S}_\mathbb{R}^0(\mathbb{R}^2) \), and that \( K_\alpha^1 \mathcal{R}_\alpha \) is its inverse; in particular for \( f \in \mathcal{S}_\mathbb{R}^0(\mathbb{R}^2) \), we have

\[
K_\alpha^1 \mathcal{R}_\alpha \mathcal{R}_\alpha (f) = f.
\]

(4.25)

(ii) Let \( f \in \mathcal{S}_\mathbb{R}^0(\mathbb{R}^2) \). From (i), we have

\[
K_\alpha^1 \mathcal{R}_\alpha \mathcal{R}_\alpha (f) = f.
\]

(4.26)

Let us put \( g = \mathcal{R}_\alpha(f) \), then \( g \in \mathcal{S}_\mathbb{R}^0(\mathbb{R}^2) \), and we have

\[
\mathcal{R}_\alpha^{-1}(g) = K_\alpha^1 \mathcal{R}_\alpha (g),
\]

(4.27)

and from Lemma 3.6, it follows that

\[
\mathcal{R}_\alpha^{-1}(g) = \mathcal{L}_\alpha^{-1} \left( \frac{\pi}{2^{2a+1} \Gamma^2(\alpha + 1)} (\mu^2 + \lambda^2)^{a/2} |\mu| \mathcal{F}_\alpha (g) \right),
\]

\[
\mathcal{R}_\alpha^{-1} \mathcal{F}_\alpha^{-1}(g) = \mathcal{F}_\alpha^{-1} \left( \frac{\pi}{2^{2a+1} \Gamma^2(\alpha + 1)} (\mu^2 + \lambda^2)^{a/2} |\mu| \mathcal{F}_\alpha (g) \right) = K_\alpha^2(g),
\]

(4.28)

which gives

\[
f = \mathcal{F}_\alpha K_\alpha^2 \mathcal{F}_\alpha (f).
\]

(4.29)

\[\square\]

4.2. The expressions of the operators \( K_\alpha^1 \) and \( K_\alpha^2 \). In the previous subsection, we have defined the operators \( K_\alpha^1 \) and \( K_\alpha^2 \) in terms of Fourier transforms \( \Lambda_\alpha \) and \( \mathcal{F}_\alpha \). Here, we will give nice expressions of these operators using fractional powers of partial differential operators. For this, we need the following inevitable notations.

(i) \( \mathcal{E}_\alpha(\mathbb{R}) \) is the space of even infinitely differentiable functions on \( \mathbb{R} \).

(ii) \( \mathcal{S}_\alpha(\mathbb{R}) \) is the subspace of \( \mathcal{E}_\alpha(\mathbb{R}) \), consisting of functions rapidly decreasing together with all their derivatives.

(iii) \( \mathcal{S}'_\alpha(\mathbb{R}) \) is the space of even tempered distributions on \( \mathbb{R} \).
Inversion formulas for Riemann-Liouville transform

(iv) $S^\omega_\ast(\mathbb{R}^2)$ is the space of tempered distributions on $\mathbb{R}^2$, even with respect to the first variable.

Each of these spaces is equipped with its usual topology.

(i) For $a \in \mathbb{R}$, $a \geq -1/2$, $d\omega_a(r)$ is the measure defined on $[0, +\infty[$ by

$$d\omega_a(r) = \frac{1}{2^a \Gamma(a+1)} r^{2a+1} dr. \tag{4.30}$$

(ii) $\ell_a$ is the Bessel operator defined on $]0, +\infty[$ by

$$\ell_a = \frac{d^2}{dr^2} + \frac{2a + 1}{r} \frac{d}{dr}, \quad a \geq -\frac{1}{2}. \tag{4.31}$$

(iii) For an even measurable function $f$ on $\mathbb{R}$, $T^\omega_{f}$ is the element of $\mathcal{S}_\ast(\mathbb{R})$, defined by

$$\langle T^\omega_{f}, \varphi \rangle = \int_0^{+\infty} f(r) \varphi(r) d\omega_a(r), \quad \varphi \in \mathcal{S}_\ast(\mathbb{R}). \tag{4.32}$$

(iv) For a measurable function $g$ on $\mathbb{R}^2$, even with respect to the first variable, $T^m_{g}$ (resp., $T^v_{g}$) is the element of $\mathcal{S}_\ast'(\mathbb{R}^2)$, defined by

$$\langle T^m_{g}, \varphi \rangle = \int_{\mathbb{R}} \int_0^{+\infty} g(r, x) \varphi(r, x) dm(r, x), \quad \varphi \in \mathcal{S}_\ast(\mathbb{R}^2), \tag{4.33}$$

where $dv$ and $dm$ are the measures defined by the relations (3.1) and (3.26).

Definition 4.7. (i) The translation operator $\tau^a_r$, $r \in \mathbb{R}$, associated with Bessel operator $\ell_a$ is defined on $\mathcal{S}_\ast(\mathbb{R})$ by the following. For all $s \in \mathbb{R}$,

$$\tau^a_r f(s) = \begin{cases} \Gamma(a+1) \sqrt{\pi} (a+1/2) \int_0^\pi f(\sqrt{r^2 + s^2 + 2rs \cos \theta}) \sin^{2a} \theta d\theta & \text{if } a > -\frac{1}{2}, \\ f(r+s) + f(|r-s|) & \text{if } a = -\frac{1}{2}. \end{cases} \tag{4.34}$$

(ii) The convolution product of $f \in \mathcal{S}_\ast(\mathbb{R})$ and $T \in \mathcal{S}'_\ast(\mathbb{R})$ is defined by

$$\forall r \in \mathbb{R}, \quad T \ast_a f(r) = \langle T, \tau^a_r f \rangle. \tag{4.35}$$

(iii) The Fourier Bessel transform is defined on $\mathcal{S}_\ast(\mathbb{R})$ by

$$\forall \mu \in \mathbb{R}, \quad F_a(f)(\mu) = \int_0^{+\infty} f(r) j_a(\mu r) d\omega_a(r), \tag{4.36}$$

and on $\mathcal{S}'_\ast(\mathbb{R})$ by

$$\forall \varphi \in \mathcal{S}_\ast(\mathbb{R}), \quad \langle F_a(T), \varphi \rangle = \langle T, F_a(\varphi) \rangle. \tag{4.37}$$
We have the following properties (we refer to [19]).

(i) $F_a$ is an isomorphism from $\mathcal{S}_a(\mathbb{R})$ (resp., $\mathcal{S}'_a(\mathbb{R})$) onto itself, and we have
\[ F_a^{-1} = F_a. \] (4.38)

(ii) For $f \in \mathcal{S}_a(\mathbb{R})$, and $r \in \mathbb{R}$, $\tau^a_r f$ belongs to $\mathcal{S}_a(\mathbb{R})$, and we have
\[ F_a(\tau^a_r f)(\mu) = j_a(r\mu)F_a(f)(\mu). \] (4.39)

(iii) For $f \in \mathcal{S}_a(\mathbb{R})$ and $T \in \mathcal{S}'_a(\mathbb{R})$, the function $T \ast_a f$ belongs to $\mathcal{E}_a(\mathbb{R})$, and is slowly increasing, moreover
\[ F_a(T^{\alpha_a}_{T \ast_a f}) = F_a(f)F_a(T). \] (4.40)

In the following, we will define the fractional powers of Bessel operator and the Laplacian operator defined on $\mathbb{R}^2$ by
\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial x^2} \] (4.41)
that we use to give simple expressions of $K^1_a$ and $K^2_a$.

In [16], the author has proved that the mappings
\[ z \mapsto T^{\alpha_a}_{|z|^2}, \quad z \mapsto T^{\alpha_a}_{(2^{\alpha + 1} \Gamma(z/2 + a + 1)/\Gamma(-z/2))|z|^2} \] defined initially for $-2(a + 1) < \Re e(z) < 0$, can be extended to a valued functions on $\mathcal{S}'_a(\mathbb{R})$, analytic on $\mathbb{C} \setminus \{-2(k + a), k \in \mathbb{N}^* \}$, and we have
\[ T^{\alpha_a}_{|z|^2} = F_a \left( T^{\alpha_a}_{(2^{\alpha + 1} \Gamma(z/2 + a + 1)/\Gamma(-z/2))|z|^2} \right). \] (4.43)

**Definition 4.8.** For $z \in \mathbb{C} \setminus \{-2(k + a), k \in \mathbb{N}^* \}$, the fractional power of Bessel operator $\ell_a$ is defined on $\mathcal{S}_a(\mathbb{R})$ by
\[ (-\ell_a)^z f(r) = \left( T^{\alpha_a}_{(2^{\alpha + 1} \Gamma(z + \alpha + 1)/\Gamma(-z))|r|^2} \right) *_a f(r). \] (4.44)

From the relations (4.40) and (4.43), we deduce that for $f \in \mathcal{S}_a(\mathbb{R})$ and $z \in \mathbb{C} \setminus \{-2(k + a), k \in \mathbb{N}^* \}$, we have
\[ F_a \left( T^{\alpha_a}_{(-\ell_a)^z f} \right) = F_a(f)T^{\alpha_a}_{|z|^2}. \] (4.45)

On the other hand, from [8, 10], we deduce that the mappings
\[ z \mapsto T^{m}_{(r^2 + x^2)^z}, \quad T^{m}_{(2^{2z+1} \Gamma(z+1)\Gamma(\alpha+1)/\Gamma(-z))(r^2 + x^2)^{-z-1}}, \] defined initially for $-1 < \Re e(z) < 0$, can be extended to a valued functions in $\mathcal{S}'_a(\mathbb{R}^2)$, analytic on $\mathbb{C} \setminus \{-k, k \in \mathbb{N}^* \}$, and we have
\[ T^{m}_{(r^2 + x^2)^z} = \Lambda_a \left( T^{m}_{(2^{2z+1} \Gamma(z+1)\Gamma(\alpha+1)/\Gamma(-z))(r^2 + x^2)^{-z-1}} \right), \] (4.47)
where \( \Lambda_\alpha \) is defined on \( \mathcal{S}_\alpha^* (\mathbb{R}^2) \) by
\[
\langle \Lambda_\alpha (T), \varphi \rangle = \langle T, \Lambda_\alpha (\varphi) \rangle, \quad \varphi \in \mathcal{S}_\alpha (\mathbb{R}^2),
\]
and \( \Lambda_\alpha (\varphi) \) is given in Lemma 3.6.

**Definition 4.9.** For \( z \in \mathbb{C}\setminus \{ -k, k \in \mathbb{N}^* \} \), the fractional power of the Laplacian operator \( \Delta \) is defined on \( \mathcal{S}_\alpha^* (\mathbb{R}^2) \) by
\[
(-\Delta)^z f(r, x) = \left( T^m_{(1/\pi)(2^{2z+1}\Gamma(z+1)/(\Gamma(-z)(s^2+y^2)^{-z-1})} \right)(r, x),
\]
where
(i) \( * \) is the usual convolution product defined by
\[
T * f(r, x) = \langle T, \sigma(r, x) f \rangle, \quad T \in \mathcal{S}_\alpha^* (\mathbb{R}^2), \quad f \in \mathcal{S}_\alpha (\mathbb{R}^2);
\]
(ii)
\[
\sigma(r, x) f(s, y) = \frac{1}{2} [f(r + s, y - x) + f(r - s, y - x)], \quad f \in \mathcal{S}_\alpha (\mathbb{R}^2).
\]

It is well known that for \( f \in \mathcal{S}_\alpha^* (\mathbb{R}^2) \) and \( T \in \mathcal{S}_\alpha^* (\mathbb{R}^2) \), the function \( T * f \) belongs to \( \mathcal{C}_\alpha (\mathbb{R}^2) \) and is slowly increasing, and we have
\[
\Lambda_\alpha (T^m T * f) = \Lambda_\alpha (f) \Lambda_\alpha (T),
\]
thus from the relations (4.47) and (4.52), we deduce that for \( f \in \mathcal{S}_\alpha^* (\mathbb{R}^2) \) and \( z \in \mathbb{C}\setminus \{ -k, k \in \mathbb{N}^* \} \),
\[
\Lambda_\alpha \left( T^m_{\sqrt{2\pi}^2 \Gamma(z+1)(-\Delta)^z} f \right) = \Lambda_\alpha (f) T^m_{(\rho^2+\chi^2)^z}.
\]

**Theorem 4.10.** The operator \( K_\alpha^1 \) defined in Theorem 4.5 can be written as
\[
K_\alpha^1 (f) = \frac{\pi}{2^{2\alpha+1}\Gamma^2(\alpha+1)} \left( -\frac{\partial^2}{\partial r^2} \right)^{1/2} (-\Delta)\alpha f,
\]
where
\[
\left( -\frac{\partial^2}{\partial r^2} \right)^{1/2} f(r, x) = \left( -\ell_{-1/2} \right)^{1/2} (f(\cdot, x)) (r).
\]

**Proof.** Let \( f \in \mathcal{S}_\alpha,\mathcal{B} (\mathbb{R}^2) \). Using Fubini’s theorem, we get for every \( \varphi \in \mathcal{S}_\alpha (\mathbb{R}^2) \) the following:
\[
\langle \Lambda_\alpha \left( T^m_{(-\partial^2/\partial r^2)^{1/2}} f \right), \varphi \rangle
= \frac{1}{2^{2\alpha+2}\Gamma^2(\alpha+1)} \int_{\mathbb{R}} \int_{0}^{+\infty} \left( T^m_{(-\ell_{-1/2}) (f(\cdot, x)) \ell_{-1/2} (\varphi(\cdot, y))} \right) \times \exp(-ixy) dx dy
\]

(4.56)
and by the relation (4.45), we obtain

\[
\langle \Lambda_{\alpha} \left( T_{(-\partial^2/\partial r^2)^{1/2} f} \right), \varphi \rangle = \frac{1}{2^{2\alpha+2}\Gamma^2(\alpha + 1)} \int_{\mathbb{R}} \int_0^{+\infty} \int F_{-1/2}(f(\cdot, x)) T_{|r|}^{\alpha-1/2} \varphi(\cdot, y) \times \exp(-ixy)dxdy,
\]

(4.57)

which involves that

\[
\langle \Lambda_{\alpha} \left( T_{(-\partial^2/\partial r^2)^{1/2} f} \right), \varphi \rangle = \int_{\mathbb{R}} \int_0^{+\infty} r \Lambda_{\alpha}(f)(r, y) \varphi(r, y) dm(r, y),
\]

(4.58)

this shows that

\[
\Lambda_{\alpha} \left( T_{(-\partial^2/\partial r^2)^{1/2} f} \right) = T_{|r|} \Lambda_{\alpha}(f),
\]

(4.59)

Now, from Lemma 4.1, we deduce that the function

\[
(\mu, \lambda) \mapsto |\mu|\Lambda_{\alpha}(f)(\mu, \lambda)
\]

belongs to the subspace \(N\). Then, from the relation (4.59), it follows that the function \((-\partial^2/\partial r^2)^{1/2} f\) belongs to the subspace \(\mathcal{F}_{*,0}(\mathbb{R}^2)\), and we have

\[
\forall (\mu, \lambda) \in \mathbb{R}^2, \quad \Lambda_{\alpha} \left( -\partial^2/\partial r^2 \right)^{1/2} f(\mu, \lambda) = |\mu|\Lambda_{\alpha}(\mu, \lambda).
\]

(4.61)

By the same way, and using the relation (4.53), we deduce that for every \(f \in \mathcal{F}_{*,0}(\mathbb{R}^2)\), the function \((-\Delta)^{a} f\) belongs to the subspace \(\mathcal{F}_{*,0}(\mathbb{R}^2)\), and we have that for all \((\mu, \lambda) \in \mathbb{R}^2\),

\[
\Lambda_{\alpha} \left( \sqrt{2\pi^2} \Gamma(\alpha + 1)(-\Delta)^{a} f \right)(\mu, \lambda) = (\mu^2 + \lambda^2)^{a} \Lambda_{\alpha}(f)(\mu, \lambda).
\]

(4.62)

Hence, the theorem follows from the relations (4.61) and (4.62).

\(\square\)

**Definition 4.11.** Let \(a, b \in \mathbb{R}, b \geq a \geq -1/2\).

(i) The Sonine transform is the mapping defined on \(\mathcal{E}_*(\mathbb{R})\) by the following. For all \(r \in \mathbb{R}\),

\[
S_{b,a}(f)(r) = \begin{cases} 
\frac{2\Gamma(b+1)}{\Gamma(b-a)\Gamma(a+1)} \int_0^1 (1-t^2)^{b-a-1} f(rt)t^{2a+1} dt & \text{if } b > a, \\
\frac{f(r)}{2} & \text{if } b = a.
\end{cases}
\]

(4.63)

(ii) The dual transform \(S_{b,a}^*\) is the mapping defined on \(\mathcal{F}_*(\mathbb{R})\) by the following. For all \(r \in \mathbb{R}\),

\[
S_{b,a}^*(f)(r) = \begin{cases} 
\frac{2\Gamma(b+1)}{\Gamma(b-a)\Gamma(a+1)} \int_r^{+\infty} (t^2 - r^2)^{b-a-1} f(t) dt & \text{if } b > a, \\
\frac{f(r)}{2} & \text{if } b = a.
\end{cases}
\]

(4.64)
Then, we have the following.

(i) The Sonine transform is an isomorphism from $\mathcal{E}_a(\mathbb{R})$ onto itself.

(ii) The dual Sonine transform is an isomorphism from $\mathcal{F}^*_a(\mathbb{R})$ onto itself.

(iii) For $f \in \mathcal{E}_a(\mathbb{R})$, $f$ bounded, and $g \in \mathcal{F}^*_a(\mathbb{R})$, we have

$$
\int_0^{+\infty} S_{b,a}(f)(r)g(r)r^{2b+1}dr = \int_0^{+\infty} f(r)^t S_{b,a}(g)(r)r^{2a+1}dr.
$$

(iv) $j_b = S_{b,a}(j_a)$.

(v)

$$
F_b = \frac{\Gamma(a+1)}{2^{b-a}\Gamma(b+1)} F_a \circ S_{b,a}.
$$

For more details, we refer to [18, 20, 21].

We denote the following.

(i) For $T \in \mathcal{F}^*_a(\mathbb{R}^2)$, $\varphi \in \mathcal{F}^*_a(\mathbb{R}^2)$,

$$
\langle S_{a,0}(T), \varphi \rangle = \langle T, \psi \rangle,
$$

with $\psi(r,x) = {}^\ast S_{a,0}(\varphi(\cdot,x))(r)$.

(ii) For all $(r,x) \in \mathbb{R}^2$,

$$
T\#\varphi(r,x) = \langle T, \overline{T}(r,-x)\varphi \rangle,
$$

where $\overline{T}(r,x)$ is the translation operator given by Definition 3.1.

(iii) $\widetilde{\varphi}_a$ is the mapping defined on $\mathcal{F}^*_a(\mathbb{R}^2)$ by

$$
\forall \varphi \in \mathcal{F}^*_a(\mathbb{R}^2), \quad \langle \widetilde{\varphi}_a(T), \varphi \rangle = \langle T, \widetilde{\varphi}_a(\varphi) \rangle.
$$

(iv) $L_\alpha$ is the operator defined on $\mathcal{F}^*_a(\mathbb{R}^2)$ by

$$
L_\alpha f(r,x) = ( - \ell_\alpha )^{2a} ( f(\cdot,x) )(r),
$$

where $(-\ell_\alpha)^z$ is the fractional power of Bessel given by Definition 4.8.

**Theorem 4.12.** The operator $K^2_\alpha$, defined in Theorem 4.5, is given by

$$
K^2_\alpha(f)(r,x) = \frac{\pi}{2^{4a+2}\Gamma(a+1)} S_{a,0}(T)(-\Delta_2)L_\alpha(\hat{f})(r,-x), \quad f \in \mathcal{F}^0_*(\mathbb{R}^2),
$$

where

(i) $T$ is the distribution defined by

$$
\langle T, \varphi \rangle = \int_{\mathbb{R}^2} \varphi(y,y)dy;
$$

(ii) $\Delta_2$ is the operator defined in Section 2.
Proof. By the definition of $K^2_a$, and the relation (3.12), we have that for $f \in \mathcal{S}_a(\mathbb{R}^2)$,

$$K^2_a(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3\alpha+1} \Gamma^3(\alpha+1)} \int_0^{+\infty} \mu^2 (\mu^2 + \lambda^2)^{2\alpha/\lambda} \mathcal{F}_a(f)(\sqrt{\mu^2 + \lambda^2}) j_{\alpha}(r \sqrt{\mu^2 + \lambda^2}) \exp(i\lambda x) d\mu d\lambda. \tag{4.73}$$

By a change of variables, and using Fubini's theorem, we get

$$K^2_a(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3\alpha+1} \Gamma^3(\alpha+1)} \int_0^{+\infty} \int_{-\nu}^{\nu} \exp\left(i\lambda \sqrt{\nu^2 - \lambda^2}\right) \mathcal{F}_a(f)(\nu,\lambda) \frac{\exp(i\lambda x)}{\sqrt{\nu^2 - \lambda^2}} j_a(r\nu) \nu d\nu d\lambda. \tag{4.74}$$

On the other hand, for $f \in \mathcal{S}_a(\mathbb{R}^2)$, the function $L_a f$ belongs to $\mathcal{E}_a(\mathbb{R}^2)$, and is slowly increasing. Moreover, we have

$$\tilde{\mathcal{F}}_a \left( T^\nu_{L_a f} \right) = T^\nu_{|\mu|^{4\alpha} \tilde{\mathcal{F}}_a(f)}. \tag{4.75}$$

But, for $f \in \mathcal{S}_a(\mathbb{R}^2)$, the function $\tilde{\mathcal{F}}_a(f)$ belongs to the subspace $N'$; according to Theorem 4.2, we deduce that the function $L_a f$ belongs to $\mathcal{S}_a(\mathbb{R}^2)$, and we have

$$\forall (\mu,\lambda) \in \mathbb{R}^2, \quad \tilde{\mathcal{F}}_a(L_a f)(\mu,\lambda) = |\mu|^{4\alpha} \tilde{\mathcal{F}}_a(f)(\mu,\lambda). \tag{4.76}$$

This involves that

$$K^2_a(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3\alpha+1} \Gamma^3(\alpha+1)} \int_0^{+\infty} \int_{-\nu}^{\nu} \exp\left(i\lambda \sqrt{\nu^2 - \lambda^2}\right) \mathcal{F}_a(L_a f)(\nu,\lambda) \frac{\exp(i\lambda x)}{\sqrt{\nu^2 - \lambda^2}} j_a(r\nu) \nu d\nu d\lambda$$

$$= \frac{\sqrt{\pi/2}}{2^{3\alpha+1} \Gamma^3(\alpha+1)} \int_0^{+\infty} \int_{-\nu}^{\nu} \tilde{\mathcal{F}}_a((- \Delta_2) L_a f)(\nu,\lambda) \frac{\exp(i\lambda x)}{\sqrt{\nu^2 - \lambda^2}} j_a(r\nu) \nu d\nu d\lambda. \tag{4.77}$$

Since for every $f \in \mathcal{S}_a(\mathbb{R}^2)$, we have that

$$\forall (r,x), (\mu,\lambda) \in \mathbb{R}^2, \quad \tilde{\mathcal{F}}_a(T_{(r,x)} f)(\nu,\lambda) = j_a(r\nu) \exp(i\lambda x) \tilde{\mathcal{F}}_a(f)(\nu,\lambda), \tag{4.78}$$

we get

$$K^2_a(f)(r,x) = \frac{\sqrt{\pi/2}}{2^{3\alpha+1} \Gamma^3(\alpha+1)} \int_0^{+\infty} \int_{-\nu}^{\nu} \tilde{\mathcal{F}}_a(T_{(r,x)} ((- \Delta_2) L_a f)) (\nu,\lambda) \frac{\nu d\nu d\lambda}{\sqrt{\nu^2 - \lambda^2}}. \tag{4.79}$$

Using the expression of $\tilde{\mathcal{F}}_a$, we obtain

$$K^2_a(f)(r,x) = \frac{1}{2^{4\alpha+2} \Gamma(\alpha+1)} \int_0^{+\infty} \int_{-\nu}^{\nu} \left[ \int_{\mathbb{R}} \int_{0}^{+\infty} \left( T_{(r,x)} ((- \Delta_2) L_a f) (s,y) \right) d\lambda \right] \frac{d\lambda}{\sqrt{\nu^2 - \lambda^2}} dy. \tag{4.80}$$
From the fact that
\[
\int_{-\nu}^{\nu} \frac{\exp(-i\lambda y)}{\sqrt{\nu^2 - \lambda^2}} d\lambda = \pi j_0(\nu y),
\] (4.81)
and using Fubini’s theorem, we deduce that
\[
K_2^2(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^4(\alpha + 1)} \int_{\mathbb{R}} \left\{ \int_0^{+\infty} \left( 0 \right)^\alpha_L(s,y) j_\alpha(sv) \times j_0(\nu y) s^{2\alpha+1} ds dy \right\} dy
\]
(4.82)
and from the relation (4.66), we have
\[
K_3^2(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^4(\alpha + 1)} \int_{\mathbb{R}} \left\{ \int_0^{+\infty} F_0 \circ \left( \left( -\Delta_L \right) f \right)(\cdot,y) (\nu) \times j_0(\nu y) dy \right\} dy,
\] (4.83)
and the relation (4.38) implies that
\[
K_3^2(f)(r,x) = \frac{\pi}{2^{4\alpha+2}\Gamma^4(\alpha + 1)} \int_{\mathbb{R}} F_0 \circ \left( \left( -\Delta_L \right) f \right)(\nu) dy.
\] (4.84)

4.3. Plancherel theorem for $^t\mathcal{R}_a$

**Proposition 4.13.** The operator $K_3^3$ defined by
\[
K_3^3(f) = \pi \left( -\frac{\partial^2}{\partial r^2} \right)^{1/4} \left( -\Delta \right)^{\alpha/2} f
\] (4.85)
is an isomorphism from $\mathcal{F}_{*,0}(\mathbb{R}^2)$ onto itself, where
\[
\left( -\frac{\partial^2}{\partial r^2} \right)^{1/4} f(r,x) = \left( -\ell_{-1/2} \right)^{1/4} (f(\cdot,x))(r).
\] (4.86)

**Proof.** Let $f \in \mathcal{F}_{*,0}(\mathbb{R}^2)$. From the relations (4.45) and (4.53), we deduce that for all $(\mu,\lambda) \in \mathbb{R}^2$,
\[
\sqrt{\mu}(|\mu^2 + \lambda^2|)^{\alpha/2} \Lambda_a(f)(\mu,\lambda) = \Lambda_a \left( \sqrt{2\pi 2^\alpha \Gamma(\alpha + 1)} \left( -\frac{\partial^2}{\partial r^2} \right)^{1/4} \left( -\Delta \right)^{\alpha/2} f \right)(\mu,\lambda),
\] (4.87)
which implies that for all \((\mu, \lambda) \in \mathbb{R}^2\),

\[
\Lambda_\alpha(K^3_\alpha(f))(\mu, \lambda) = \sqrt{\frac{\pi}{2^\alpha \Gamma(\alpha + 1)}} |\mu|^{\alpha/2} \Lambda_\alpha(f)(\mu, \lambda).
\] (4.88)

Then, the result follows from Lemma 4.1 and Theorem 4.2.

**Proposition 4.14.** For \(g \in \mathcal{S}_0^0(\mathbb{R}^2)\), there exists the Plancherel formula

\[
\int_0^{+\infty} \int_0^{+\infty} |g(r, x)|^2 d\nu(r, x) = \int_0^{+\infty} \int_0^{+\infty} |K^3_\alpha(\mathcal{R}_\alpha(g))(r, x)|^2 dm(r, x).
\] (4.89)

**Proof.** Let \(g \in \mathcal{S}_0^0(\mathbb{R}^2)\), from Theorem 3.5 (Plancherel formula), we have

\[
\int_0^{+\infty} \int_0^{+\infty} |g(r, x)|^2 d\nu(r, x) = \int_0^{+\infty} \int_0^{+\infty} |\tilde{F}_\alpha(g)(\mu, \lambda)|^2 d\gamma(\mu, \lambda).
\] (4.90)

From the relation (3.12), Lemma 3.6, and the fact that

\[
\supp \tilde{\mathcal{S}}_\alpha(g) \subset \{(\mu, \lambda) \in \mathbb{R}^2/|\mu| \geq |\lambda|\};
\] (4.91)

we get

\[
\int_0^{+\infty} \int_0^{+\infty} |g(r, x)|^2 d\nu(r, x) = \int_0^{+\infty} \int_0^{+\infty} |\sqrt{\mu}^{\alpha/2} \Lambda_\alpha \circ t_{R^\alpha}(\mathcal{R}_\alpha(g))(\mu, \lambda)|^2 dm(\mu, \lambda).
\] (4.92)

We complete the proof by using the formula (4.88), and the fact that for every \(f \in \mathcal{S}_0^0(\mathbb{R}^2)\),

\[
\int_0^{+\infty} \int_0^{+\infty} |\Lambda_\alpha(f)(\mu, \lambda)|^2 dm(\mu, \lambda) = \frac{\pi}{2^{2\alpha+1} \Gamma^2(\alpha + 1)} \int_0^{+\infty} \int_0^{+\infty} |f(\mu, \lambda)|^2 dm(\mu, \lambda).
\] (4.93)

We denote by

(i) \(L^2_0(d\nu)\) the subspace of \(L^2(d\nu)\) consisting of functions \(g\) such that

\[
\supp \tilde{\mathcal{S}}_\alpha(g) \subset \{(\mu, \lambda) \in \mathbb{R}^2/|\mu| \geq |\lambda|\};
\] (4.94)

(ii) \(L^2(dm)\) the space of square integrable functions on \([0, +\infty[\times\mathbb{R}\) with respect to the measure \(dm(r, x)\).

**Theorem 4.15.** The operator \(K^3_\alpha \circ \mathcal{R}_\alpha\) can be extended to an isometric isomorphism from \(L^2_0(d\nu)\) onto \(L^2(dm)\).
Proof. The theorem follows from Propositions 4.13, 4.14, and the density of $\mathcal{S}_{*,0}(\mathbb{R}^2)$ (resp., $\mathcal{S}_*^0(\mathbb{R}^2)$) in $L^2(dm)$ (resp., $L^2_0(d\nu)$).

5. Transmutation operators

Proposition 5.1. The Riemann-Liouville transform and its dual satisfy the following permutation properties.

(i) For all $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$,

$$\mathcal{R}_a(\Delta_2 f) = \frac{\partial^2}{\partial r^2} \mathcal{R}_a(f), \quad \mathcal{R}_a(\Delta_1 f) = \Delta_1 \mathcal{R}_a(f). \quad (5.1)$$

(ii) For all $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$,

$$\Delta_2 \mathcal{R}_a(f) = \mathcal{R}_a\left(\frac{\partial^2 f}{\partial r^2}\right), \quad \Delta_1 \mathcal{R}_a(f) = \mathcal{R}_a(\Delta_1 f). \quad (5.2)$$

Proof. (i) We know that the operators $\Delta_1, \Delta_2, \partial^2/\partial r^2$, and $\mathcal{R}_a$ are continuous mappings from $\mathcal{S}_{*,0}(\mathbb{R}^2)$ onto itself. Then, by applying the usual Fourier transform $\Lambda_a$, we have

$$\Lambda_a\left(\mathcal{R}_a(\Delta_2 f)\right)(\mu, \lambda) = -\mu^2 \Lambda_a \circ \mathcal{R}_a(f)(\mu, \lambda) = \Lambda_a\left(\frac{\partial^2}{\partial r^2} \mathcal{R}_a(f)\right)(\mu, \lambda),$$

$$\Lambda_a(\Delta_1 \mathcal{R}_a f)(\mu, \lambda) = i\lambda \Lambda_a\left(\mathcal{R}_a(f)\right)(\mu, \lambda) = \Lambda_a\left(\mathcal{R}_a(\Delta_1 f)\right)(\mu, \lambda). \quad (5.3)$$

Consequently, (i) follows from the fact that $\Lambda_a$ is an isomorphism from $\mathcal{S}_{*,0}(\mathbb{R}^2)$ onto itself.

(ii) We obtain the result from (i), Lemma 2.4, and the fact that for $f \in \mathcal{S}_{*,0}(\mathbb{R}^2)$, and $g \in \mathcal{D}_{*,0}(\mathbb{R}^2)$,

$$\int_\mathbb{R} \int_0^{+\infty} \Delta_2 f(r,x)g(r,x) d\nu(r,x) = \int_\mathbb{R} \int_0^{+\infty} f(r,x)\Delta_2 g(r,x) d\nu(r,x). \quad (5.4)$$

Theorem 5.2. (i) The Riemann-Liouville transform $\mathcal{R}_a$ is a transmutation operator of

$$\frac{\partial^2}{\partial r^2}, \Delta_1 \quad \text{into} \quad \Delta_2, \Delta_1 \quad (5.5)$$

from

$$\mathcal{S}_{*,0}(\mathbb{R}^2) \quad \text{onto} \quad \mathcal{S}_*^0(\mathbb{R}^2). \quad (5.6)$$
(ii) The dual transform $\mathcal{R}_\alpha$ is a transmutation operator of
\[ \Delta_2, \Delta_1 \text{ into } \frac{\partial^2}{\partial r^2}, \Delta_1 \]
from
\[ \mathcal{F}^0_*(\mathbb{R}^2) \text{ onto } \mathcal{F}_{*,0}(\mathbb{R}^2). \]

This theorem follows from Proposition 5.1 and the fact that $\mathcal{R}_\alpha$ is an isomorphism from $\mathcal{F}_{*,0}(\mathbb{R}^2)$ onto $\mathcal{F}^0_*(\mathbb{R}^2)$ and $\mathcal{R}_\alpha$ is an isomorphism from $\mathcal{F}^0_*(\mathbb{R}^2)$ onto $\mathcal{F}_{*,0}(\mathbb{R}^2)$.

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Inversion formulas for Riemann-Liouville transform


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