We study convergences of Mann and Ishikawa iteration processes for mappings of asymptotically quasi-nonexpansive type in Banach spaces.

2000 Mathematics Subject Classification: 47H09, 47H10.

1. Introduction and preliminaries. Let $D$ be a nonempty subset of a real Banach space $X$ and $T : D \to D$ a nonlinear mapping. The mapping $T$ is said to be asymptotically quasi-nonexpansive (see [5]) if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} k_n = 0$ such that

$$\|T^n x - p\| \leq (1 + k_n) \|x - p\|$$

(1.1)

for all $x \in D$, $p \in F(T)$, and $n \in \mathbb{N}$. The mapping $T$ is said to be asymptotically nonexpansive (see [3]) if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} k_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + k_n) \|x - y\|$$

(1.2)

for all $x, y \in D$ and $n \in \mathbb{N}$. The mapping $T$ is said to be a mapping of asymptotically nonexpansive type [4] if

$$\limsup_{n \to \infty} \sup_{x \in D} (\|T^n x - T^n y\| - \|x - y\|) \leq 0$$

(1.3)

for any $y \in D$.


Recently, Liu [5] extended results of [2, 7] and gave the necessary and sufficient conditions for Ishikawa iterative sequence to converge to fixed points of asymptotically quasi-nonexpansive mappings.
First, we introduce the concept of class of mappings of asymptotically quasi-nonexpansive type: the mapping $T$ is said to be a mapping of asymptotically quasi-nonexpansive type if $F(T) \neq \emptyset$ and

\[
\limsup_{n \to \infty} \sup_{x \in D} (\|T^n x - p\| - \|x - p\|) \leq 0 \quad \text{for any } p \in F(T). \quad (1.4)
\]

**Remark 1.1.** If $T$ is a mapping of asymptotically nonexpansive type with $F(T) \neq \emptyset$, then $T$ is a mapping of asymptotically quasi-nonexpansive type.

**Remark 1.2.** If $D$ is bounded and $T$ is an asymptotically quasi-nonexpansive mapping, then $T$ is a mapping of asymptotically quasi-nonexpansive type. In fact, if $T$ is an asymptotically quasi-nonexpansive mapping, then there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} k_n = 0$ such that

\[
\|T^n x - p\| \leq (1 + k_n) \|x - p\| \quad (1.5)
\]

for all $x \in D$, $p \in F(T)$, and $n \in \mathbb{N}$, which implies

\[
\sup_{x \in D} \{|\|T^n x - T^n y\| - \|x - y\|\} \leq k_n \cdot \text{diam } D \quad (1.6)
\]

for any $y \in F(T)$ and $n \in \mathbb{N}$. Hence

\[
\limsup_{n \to \infty} \sup_{x \in D} (\|T^n x - T^n y\| - \|x - y\|) \leq 0 \quad \text{for any } y \in F(T). \quad (1.7)
\]

We observe from Remarks 1.1 and 1.2 that the class of mappings of asymptotically nonexpansive type is an intermediate class between the class of mappings of asymptotically quasi-nonexpansive type and that of mappings of asymptotically nonexpansive type with nonempty fixed-point sets. Let

\[C_1 = \{T : T : D \to D \text{ is a nonexpansive mapping}\},\]

\[C_2 = \{T : T : D \to D \text{ is a quasi-nonexpansive mapping}\},\]

\[C_3 = \{T : T : D \to D \text{ is an asymptotically nonexpansive mapping}\},\]

\[C_4 = \{T : T : D \to D \text{ is an asymptotically quasi-nonexpansive mapping}\},\]

\[C_5 = \{T : T : D \to D \text{ is a mapping of asymptotically nonexpansive type}\},\]

\[C_6 = \{T : T : D \to D \text{ is a mapping of asymptotically quasi-nonexpansive type}\}. \quad (1.8)
\]

Then we have the following implications:

\[
\begin{array}{c}
C_1 \quad \Rightarrow \quad C_2 \\
\downarrow \quad \downarrow \\
C_3 \quad \Rightarrow \quad C_4 \\
\downarrow \quad \downarrow \\
C_5 \quad \Rightarrow \quad C_6.
\end{array} \quad (1.9)
\]
In this paper, we are mainly interested in the problem of approximation of fixed points of the more general class of mappings of asymptotically quasi-nonexpansive type than that of asymptotically quasi-nonexpansive mappings. The purpose of this paper is to continue discussion concerning convergence of Mann and Ishikawa iteration processes for mappings of asymptotically quasi-nonexpansive type in Banach spaces. We give necessary and sufficient conditions for the Mann and Ishikawa iteration processes to converge to fixed points of mappings of asymptotically quasi-nonexpansive type. Further, we obtain extensions of various results obtained quite recently by Deng [1], Ghosh and Denath [2], Liu [5], and Tan and Xu [9, 10] to more general types of space as well as families of operators.

We say that a Banach space $X$ satisfies Opial’s condition [6] if, for each sequence $\{x_n\}$ in $X$ weakly convergent to a point $x$ and for all $y \neq x$,

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|. \quad (1.10)$$

The examples of Banach spaces which satisfy Opial’s condition are Hilbert spaces, and all $L^p[0,2\pi]$ with $1 \neq p > 2$ fail to satisfy Opial’s condition [6].

Let $D$ be a nonempty closed convex subset of a Banach space $X$. Then $I - T$ is demiclosed at zero if, for any sequence $\{x_n\}$ in $D$, condition $x_n \to x$ weakly and $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$ implies $(I - T)x = 0$.

2. Main results. In this section, we establish some weak and strong convergences for mappings of asymptotically quasi-nonexpansive type in Banach spaces.

**Lemma 2.1.** Let $D$ be a nonempty subset of a normed space $X$ and let $T : D \to E$ be a mapping of asymptotically quasi-nonexpansive type. For two given real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,1]$, let a sequence $\{x_n\}$ in $D$ be defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \quad y_n = (1 - \beta_n)x_n + \beta_n Tx_n, \quad n = 1,2,\ldots. \quad (2.1)$$

If $p$ is a fixed point of $T$, then

(a) $\|x_{n+1} - p\| \leq \|x_n - p\| + (1 + \beta_n)\sup_{x \in D}(\|T^n x - p\| - \|x - p\|), \quad n = 1,2,\ldots,$

(b) $\lim_{n \to \infty} \|x_n - p\|$ exists.

**Proof.** Let $p$ be a fixed point of $T$.

(a) From (2.1), we have

$$\|x_{n+1} - p\| \leq (1 - \alpha_n)\|x_n - p\| + \alpha_n\|T^n y_n - p\|$$

$$\leq (1 - \alpha_n)\|x_n - p\| + \alpha_n(\|T^n y_n - p\| - \|y - p\|) + \alpha_n\|y_n - p\|$$
\[ (1 - \alpha_n) \| x_n - p \| + \| T^i y_n - p \| - \| y_n - p \| \\ + \alpha_n ((1 - \beta_n) \| x_n - p \| + \beta_n \| T^n x_n - p \|) \leq \| x_n - p \| + \| T^n y_n - p \| - \| y_n - p \| \\ + \beta_n (\| T^n x_n - p \| - \| x_n - p \|) \leq \| x_n - p \| + (1 + \beta_n) \sup_{x \in D} (\| T^n x - p \| - \| x - p \|). \]
\[(2.2)\]

(b) For \( m, n \in \mathbb{N} \), we have
\[ \| x_{n + m} - p \| \leq \| x_{n + m - 1} - p \| + 2 \sup_{x \in D} (\| T^{n + m - 1} x - p \| - \| x - p \|) \leq \| x_{n + m - 1} - p \| + 2 \sup_{x \in D} (\| T^m x - p \| - \| x - p \|) \leq \| x_{n + m - 2} - p \| + 4 \sup_{x \in D} (\| T^m x - p \| - \| x - p \|) \leq \cdots \leq \| x_n - p \| + 2n \sup_{x \in D} (\| T^m x - p \| - \| x - p \|). \]
\[(2.3)\]

Hence, for \( n \in \mathbb{N} \),
\[ \limsup_{m \to \infty} \| x_m - p \| \leq \| x_n - p \| + 2n \limsup_{m \to \infty} \sup_{x \in D} (\| T^m x - p \| - \| x - p \|) \leq \| x_n - p \|. \]
\[(2.4)\]

It follows that
\[ \limsup_{m \to \infty} \| x_m - p \| \leq \liminf_{n \to \infty} \| x_n - p \|. \]
\[(2.5)\]

Thus \( \lim_{n \to \infty} \| x_n - p \| \) exists.

**Lemma 2.2.** Let \( D \) and \( T \) be as in Lemma 2.1. For a given real sequence \( \{ \alpha_n \} \) in \([0, 1]\), let a sequence \( \{ x_n \} \) in \( D \) be defined by
\[ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, \quad n = 1, 2, \ldots \]
\[(2.6)\]

If \( p \) is a fixed point of \( T \), then
(a) \( \| x_{n+1} - p \| \leq \| x_n - p \| + \sup_{x \in D} (\| T^n x - p \| - \| x - p \|), \quad n = 1, 2, \ldots \)

(b) \( \lim_{n \to \infty} \| x_n - p \| \) exists.

**Theorem 2.3.** Let \( X \) be a Banach space which satisfies Opial’s condition and let \( D \) be a weakly compact subset of \( X \). Let \( T \) and \( \{ x_n \} \) be as in Lemma 2.1. Suppose that \( T \) has a fixed point, \( I - T \) is demiclosed at zero, and \( \{ x_n \} \) is an approximating fixed-point sequence for \( T \), that is, \( \lim_{n \to \infty} \| x_n - T x_n \| = 0 \). Then \( \{ x_n \} \) converges weakly to a fixed point of \( T \).

**Proof.** First, we show that \( \omega_w(x_n) \subset F(T) \). Let \( x_{n_k} \rightharpoonup x \) weakly. By assumption, we have \( \lim_{n \to \infty} \| x_n - T x_n \| = 0 \). Since \( I - T \) is demiclosed at zero,
$x \in F(T)$. By Opial's condition, $\{x_n\}$ possesses only one weak limit point, that is, $\{x_n\}$ converges weakly to a fixed point of $T$.

**Theorem 2.4.** Let $X$ be a Banach space which satisfies Opial's condition and let $D$ be a weakly compact subset of $X$. Let $T$ and $\{x_n\}$ be as in Lemma 2.2. Suppose that $T$ has a fixed point, $I - T$ is demiclosed at zero, and $\{x_n\}$ is an approximating fixed-point sequence for $T$, that is, $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. Then $\{x_n\}$ converges weakly to a fixed point of $T$.

**Remark 2.5.** Theorem 2.3 improves Theorem 2 of Deng [1] for mappings of asymptotically quasi-nonexpansive type. Theorem 2.4 generalizes Theorem 2.1 of Schu [8].

**Theorem 2.6.** Let $D$ be a closed subset of Banach space, let $T : D \to D$ be a mapping of asymptotically quasi-nonexpansive type, and $F(T)$ be nonempty closed set. For two given real sequences $\{\alpha_n\}$ and $\{\beta\}$ in $[0, 1]$, let the Ishikawa iterative sequence $\{x_n\}$ in $D$ be defined by (2.1). Then $\{x_n\}$ converges strongly to a fixed point of $T$ if and only if $\liminf_n d(x_n, F(T)) = 0$.

**Proof.** Let $\{x_n\}$ converge strongly to a point $z \in F(T)$. Then $\lim_n d(x_n, F(T)) = 0$. Conversely, suppose $\liminf_n d(x_n, F(T)) = 0$. From Lemma 2.1(a),

$$\|x_{n+1} - p\| \leq \|x_n - p\| + 2 \sup_{x \in D} (\|T^nx - p\| - \|x - p\|)$$

for any $n \in \mathbb{N}$ and $p \in F(T)$. Since $T$ is a mapping of asymptotically quasi-nonexpansive type, we have

$$\limsup_n \left\{ \sup_{k \geq n} \left( \sup_{x \in D} (\|T^kx - p\| - \|x - p\|) \right) \right\} \leq 0. \tag{2.8}$$

Hence, there exists a positive integer $n_0$ and a sequence $\{a_n\}$ of positive real numbers with $\lim_n a_n = 0$ such that

$$\sup_{k \geq n} \left\{ \sup_{x \in D} (\|T^kx - p\| - \|x - p\|) \right\} \leq a_n \tag{2.9}$$

for any $n \geq n_0$. Without loss of generality, we can assume that $a_n = 1/2n^2$. Hence,

$$\sup_{k \geq n} \left\{ \sup_{x \in D} (\|T^kx - p\| - \|x - p\|) \right\} \leq \frac{1}{2n^2} \tag{2.10}$$

for any $n \geq n_0$. It follows from (2.7) that

$$\|x_{n+1} - p\| \leq \|x_n - p\| + \frac{1}{n^2} \tag{2.11}$$
for all $n \geq n_0$, that is,
\[ d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + \frac{1}{n^2} \]  
\hspace{1cm} (2.12)
for all $n \geq n_0$. Hence for $n, m \geq n_0$, we have
\[ d(x_{n+m}, F(T)) \leq d(x_n, F(T)) + \sum_{i=n}^{n+m-1} \frac{1}{i^2}. \]  
\hspace{1cm} (2.13)
Using [10, Lemma 1, page 303], we obtain that $\lim_n d(x_n, F(T))$ exists, and it follows from $\liminf_n d(x_n, F(T)) = 0$ that $\lim_n d(x_n, F(T)) = 0$. Thus, $\lim_n d(x_n, F(T)) = 0$. For each $\varepsilon > 0$, there exists a natural number $m_0$ such that
\[ d(x_n, F(T)) < \frac{\varepsilon}{3} \]  
\hspace{1cm} (2.14)
for all $n \geq m_0$. Then there exists a $p' \in F(T)$ such that $d(x_n, p') < \varepsilon/2$ for all $n \geq m_0$. If $n, m \geq m_0$, then
\[ d(x_n, x_m) \leq d(x_n, p') + d(p', x_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]  
\hspace{1cm} (2.15)
This shows that \( \{x_n\} \) is a Cauchy sequence in \( D \). Let $\lim_n x_n = v \in D$. Since $F(T) \subset D$ is closed and $\lim_n d(x_n, F(T)) = 0$, we conclude that $v \in F(T)$. This completes the proof. \hfill \Box

As a consequence of Theorem 2.6, we obtain the following result.

**Theorem 2.7.** Let $D$ be a closed subset of Banach space, let $T : D \to D$ be a mapping of asymptotically quasi-nonexpansive type, and let $F(T)$ be a nonempty closed set. For a given sequence $\{\alpha_n\}$ in $[0, 1]$, let the Mann iterative sequence $\{x_n\}$ in $D$ be defined by (2.6). Then $\{x_n\}$ converges strongly to a fixed point of $T$ if and only if $\liminf_n d(x_n, F(T)) = 0$.

**Remark 2.8.** Theorems 2.6 and 2.7 extend corresponding results of Ghosh and Debnath [2], Liu [5], and Petryshyn and Williamson [7] from quasi-nonexpansive or asymptotically quasi-nonexpansive mapping to large class of non-Lipschitzian mappings.

**Acknowledgment.** The first author wishes to acknowledge the financial support of the Department of Science and Technology, India, made in the program year 2002–2003, Project No. SR/FTP/MS-15. The second author was supported by Korea Research Foundation Grant KRF-2000-DP0013.

**References**


Daya Ram Sahu: Department of Applied Mathematics, Shri Shankaracharya College of Engineering, Junwani, Bhilai 490 020, India

E-mail address: sahudr@rediffmail.com

Jong Soo Jung: Department of Mathematics, Dong-A University, Pusan 604-714, Korea

E-mail address: jungjs@mail.donga.ac.kr
Mathematical Problems in Engineering

Special Issue on
Time-Dependent Billiards

Call for Papers
This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>June 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>September 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

Edson Denis Leonel, Department of Statistics, Applied Mathematics and Computing, Institute of Geosciences and Exact Sciences, State University of São Paulo at Rio Claro, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

Alexander Loskutov, Physics Faculty, Moscow State University, Vorob’evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru

Hindawi Publishing Corporation
http://www.hindawi.com