SOME RESULTS ON CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS AND HADAMARD PRODUCT

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Abstract. By using a certain linear operator defined by a Hadamard product or convolution, several interesting subclasses of analytic functions in the unit disc are introduced and some unifying relationships between them are established. A variety of characterization results involving a certain functional and some general functions of hypergeometric type are investigated for these classes.

Key Words and Phrases: Analytic, Hadamard product, Hypergeometric functions univalent, starlike, convex, close-to-convex, Quasi-convex, Linear operator.

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1. INTRODUCTION. Let $A$ denote the class of the function $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_{n}z^{n}$$

which are analytic in the unit disc $E = \{z: |z| < 1\}$. A function $f \in A$ is said to be in the class $R(\beta)$ if, for $z \in E$ and $\beta > -1$,

$$\text{Re} \frac{zf'(z)}{f(z)} > -\beta$$

Also, a function $f \in A$ is said to belong to the class $V(\beta)$ if, for $z \in E$ and $\beta > -1$,

$$\text{Re} \left( \frac{(zf'(z))'}{f'(z)} \right) > -\beta .$$

We note that

$$f(z) \in R(\beta) \leftrightarrow zf'(z) \in V(\beta),$$

and $v(\beta) \subseteq R(\beta)$.

The classes $V(\beta)$ and $R(\beta)$ of analytic functions have been defined and studied in [9].

We define the following.

Let $f \in A$ and let $g \in R(\beta)$. Then $f \in T(\alpha, \beta)$ if, for $\alpha > -1$ and $z \in E$, $\text{Re} \frac{zf'(z)}{g(z)} > -\alpha$.

Also, let $f \in A$. Then $f \in T^{*}(\alpha, \beta)$ if, for $\alpha > -1$, $z \in E$ and $g \in V(\beta)$,
From (1.3) and (1.4), it is clear that

\[ f \in T^*(\alpha, \beta) \iff zf' \in T(\alpha, \beta) \]

and

\[ T^*(\alpha, \beta) \subseteq T(\alpha, \beta) \]

Let \( f_j(z) (j = 1, 2) \) in \( A \) be given by

\[ f_j(z) = \sum_{n=0}^{\infty} a_{n+1} z^{n+1} \quad (a_{j_1} = 1) \]

Then the Hadamard product (or convolution) \( f_1 * f_2(z) \) of \( f_1(z) \) and \( f_2(z) \) is defined by

\[ f_1 * f_2(z) = \sum_{n=0}^{\infty} a_{n+1} b_{n+1} z^{n+1} \quad (a_{j_1} = 1) \]

Let \( a_j (j = 1, \ldots, p) \) and \( b_j (j = 1, 2, \ldots, q) \) be complex numbers with \( \beta_j \neq 0, -1, -2, \ldots, j = 1, \ldots, q \).

Then the generalized hypergeometric function \( _pF_q \) is defined by

\[ _pF_q(z) = \sum_{n=0}^{\infty} \frac{(\alpha_1) \ldots (\alpha_p)}{(\beta_1) \ldots (\beta_q)n!} z^n \quad (p \leq q + 1) \]

where \( (\lambda)_n \) is the Pochhammer symbol defined by

\[ (\lambda)_n = \begin{cases} 1 & \text{if } n = 0 \\ \lambda(\lambda + 1) \ldots (\lambda + n - 1) & \text{if } n \in \mathbb{N} = \{1, 2, 3, \ldots\} \end{cases} \]

We now define the function \( \phi(a, c, z) \) by

\[ \phi(a, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (c = 0, -1, -2, \ldots, z \in E) \]

so that \( \phi(a, c) \) is an incomplete Beta function with

\[ \phi(a, c, z) = z_2F_1(1, a; c, z) \]

Corresponding to the function \( \phi(a, c) \), Carlson and Shaffer [2] defined a linear operator \( L(a, c) \) on \( A \) by the convolution

\[ L(a, c)f = \phi(a, c) * f \]

for \( f \in A \). Clearly, \( L(a, c) \) maps \( A \) onto itself, and \( L(c, a) \) is an inverse of \( L(a, c) \) provided that \( a \neq 0, -1, -2, \ldots \)

Furthermore, \( L(a, a) \) is the identity operator, and

\[ R(\beta) = L(1, 2)V(\beta), \quad V(\beta) = L(2, 1)R(\beta) \]

Also

\[ T(\alpha, \beta) = L(1, 2)T^*(\alpha, \beta), \quad T^*(\alpha, \beta) = L(2, 1)T(\alpha, \beta) \]

where \( \alpha > -1 \) and \( \beta > -1 \).

We can now define the classes of analytic function with which we shall be dealing.

**Definition 1.1.** A function \( f \in A \) is said to be in the class \( R(a, c; \beta) \) if \( L(a, c)f \) belongs to \( R(\beta) \) for \( \beta > -1 \), and \( f \in V(a, c; \beta) \) if, and only if, \( zf' \in R(a, c; \beta) \) for \( \beta > -1 \).

Similarly we have:
Definition 1.2. A function \( f \in A \) is said to be in the class \( T(a, c; \alpha, \beta) \) if \( L(a, c) f \in T(\alpha, \beta) \) for \( \alpha > -1 \) and \( \beta > -1 \). Further \( f \in T^*(a, c; \alpha, \beta) \) if, and only if, \( zf' \in T(a, c; \alpha, \beta) \) for \( \alpha > -1 \).

The following relations can easily be verified.

\[
V(a, c; \beta) = L(1, 2)R(a, c; \beta)
\]

\[
R(a, c, \beta) = L(2, 1)V(a, c; \beta)
\]

\[
V(\beta) = V(a, a; \beta) = L(1, 2)R(a, a; \beta)
\]

and

\[
R(\beta) = R(a, a; \beta) = L(2, 1)V(a, a; \beta)
\]

Also

\[
T^*(a, c; \alpha, \beta) = L(1, 2)T(a, c; \alpha, \beta)
\]

\[
T(a, c; \alpha, \beta) = L(2, 1)T^*(a, c; \alpha, \beta)
\]

\[
T^*(\alpha, \beta) = T^*(a, a; \alpha, \beta) = L(1, 2)T(a, a; \alpha, \beta)
\]

and

\[
T(\alpha, \beta) = T(a, a; \alpha, \beta) = L(2, 1)T^*(a, a; \alpha, \beta)
\]

We shall now connect these classes with the univalent functions. A single-valued function \( f \) is said to be \textit{univalent} in a domain \( D \) if it never takes on the same value twice. By \( S, K, S^*, C \) and \( C^* \) denote the subclasses of \( A \) which are respectively univalent, close-to-convex, starlike, convex and quasi-convex in \( E \). In [8], Robertson defined the subclasses of \( C \) and \( S^* \) by using the order of the class as follows. A function \( f \in S \) is called a \textit{convex function} of order \( \beta_i, 0 \leq \beta_i < 1 \), if and only if \( Re \left( \frac{zf' \alpha z}{z \beta} \right) > \beta_i, z \in E \). We denote this class as \( C(\beta_i) \). Also a function \( f \in S \) is called \textit{starlike function} of order \( \beta_i, 0 \leq \beta_i < 1 \) if and only if \( Re \left( \frac{zf' \alpha z}{z \beta} \right) > \beta_i, z \in E \). We call this class \( S^*(\beta_i) \).

Libera [3] introduced the terminology of order and type together in the class \( K(\alpha_i, \beta_i) \) of close-to-convex functions. A function \( f \in A \) is said to be close-to-convex of order \( \alpha_i \) type \( \beta_i, 0 \leq \alpha_i < 1; 0 \leq \beta_i < 1 \), if and only if there exists a function \( g \in S^*(\beta_i) \) such that \( Re \left( \frac{zf' \alpha z}{z \beta} \right) > \alpha_i, z \in E \). Further \( f \in C^*(\alpha_i, \beta_i) \Leftrightarrow zf' \in K(\alpha_i, \beta_i) \) we refer to [7].

Indeed from the above definitions of the various subclasses of the various subclasses of \( A \), we deduce readily the following:

\[
S^*(\beta_i) \subset S^* \subset R(\beta) \subset A,
\]

\[
C(\beta_i) \subset C \subset V(\beta) \subset R(\beta) \subset A
\]

and

\[
C^*(\alpha_i, \beta_i) \subset C^* \subset T^*(\alpha, \beta) \subset T(\alpha, \beta) \subset A,
\]

\[
K(\alpha_i, \beta_i) \subset K \subset T(\alpha, \beta) \subset A,
\]

where

\[
0 \leq \alpha_i < 1, \quad 0 \leq \beta_i < 1 \quad \text{and} \quad -1 < -\alpha_i \leq \alpha; \quad -1 < -\beta_i \leq \beta.
\]
2. MAIN RESULTS

We first state certain results which will be needed in proving our main theorems.

**Lemma 2.1.** [6] Let $\phi(u, v)$ be the complex function, $\phi : D \to C$, $D \subset C \times C$ (C-complex plane) and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\phi$ satisfies the conditions:

(i) $\phi(u, v)$ is continuous in $D$;
(ii) $(1, 0) \in D$ and $\text{Re}\{\phi(0, 1)\} > 0$;
(iii) $\text{Re}\{\phi(iu_2, v_1)\} < 0$ for all $(iu_2, v_1) \in D$ and such that $v_1 < (1 + u_2^2)/2$.

Let $h(z) = 1 + c_1 z + \ldots$ be analytic in $E$, such that $(h(z), z h'(z)) \in E$ for all $z \in E$. If $\text{Re}\{\phi(h(z), z h'(z))\} > 0 (z \in E)$, then $\text{Re}(h(z)) > 0$ for $z \in E$.

Let $l(f)$ denote a functional defined by

$$I_f(t) = \int_0^t t^{-1} f(t) \, dt$$

for $f \in A$ and for a real number $\lambda > 1$. The functional $I_f(t)$, when $\lambda \in N$, was studied by Bernardi [1], and in particular, $I_f(t)$ was considered earlier by Libera [4] and Livingston [5]. We note that $I_f(t)$ is a particular solution of the ordinary first order differential equation

$$t g'(t) + \lambda g(t) (t + 1)$$

at the point $z = t$. Also by comparing (1.9) and (2.1), we have $I_f(t) = L(\lambda + 2, \lambda + 1)$. For our next results we refer to [9].

**Theorem 2.1.** Let $g \in R(a, c; \lambda)$ and let, for $\lambda \geq \beta > -1$, $l(g)$ be defined by (2.1). The $l(g)$ is also in the class $R(a, c; \beta)$.

We shall now prove the following.

**Theorem 2.2.** Let $f \in T(a, c; \alpha, \beta)$ and let, for $\lambda \geq \alpha, \beta > -1$, $l(f)$ be defined by (2.1). Then $l(f) \in T(a, c; \alpha, \beta)$.

**Proof:** Since $f \in T(a, c; \alpha, \beta)$, there exists $g \in R(a, c; \beta)$ such that

$$\text{Re}\left\{\frac{z[L(a, c) f(z)]'}{L(a, c) g(z)}\right\} > -\alpha$$

Now, from Theorem 2.1, we know that $l(g) \in R(a, c; \beta)$. Let

$$z[L(a, c) f(z)]' = (1 + a) h(z) - \alpha,$$  \hspace{1cm} (2.2)

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

Note that

$$z[L(a, c) f(z)]' = (\lambda + 1) L(a, c) f(z) - \lambda L(a, c) l(f)$$

which readily yields

$$z[L(a, c) f(z)]'' = (\lambda + 1) z[L(a, c) f(z)]' - (\lambda + 1) z[L(a, c) f(z)]'$$

Now, differentiating both sides of (2.2) logarithmically and using (2.3) and (2.4), we obtain

$$\frac{(\lambda + 1) z[L(a, c) f(z)]'}{z[L(a, c) f(z)]'} - \frac{\lambda + 1) L(a, c) f(z)}{L(a, c) l(f)} = \frac{(1 + a) h(z)}{(1 + a) h(z) - \alpha}$$
or, equivalently,
\[
\frac{(\lambda + 1) L(a, c) g(z)}{z L(a, c) \gamma(z)} \frac{z L(a, c) f(z)}{L(a, c) g(z)} - \frac{z L(a, c) \Lambda(f)}{L(a, c) \Lambda(g)} \frac{(1 + \alpha) h(z)}{L(a, c) \Lambda} = (1 + \alpha) h(z) \frac{(1 + \alpha) h(z) - \alpha}{L(a, c) \Lambda - \alpha}
\]  

(2.5)

After simplification, and taking
\[
\frac{z L(a, c) \Lambda(g)}{L(a, c) \Lambda} = (1 + \beta) H(z) - \beta,
\]
where \(\text{Re} H(z) = h_1 > 0\) and \(\beta > -1\), we have, from (2.5),
\[
\frac{z L(a, c) f(z)}{L(a, c) g(z)} = (1 + \alpha) h(z) - \alpha + \frac{(1 + \alpha) h(z)}{(1 + \beta) H(z) - \beta + \lambda}
\]
or
\[
\frac{z L(a, c) f(z)}{L(a, c) g(z)} + \alpha = (1 + \alpha) h(z) + \frac{(1 + \alpha) h(z)}{(1 + \beta) H(z) - \beta + \lambda}
\]

(2.6)

We form the function \(\phi(u, v)\) by taking
\[
u = h(z) \quad \text{and} \quad v = z h'(z)
\]
in (2.6) as
\[
\phi(u, v) = (1 + \alpha) u + \frac{(1 + \alpha) v}{(1 + \beta) H(z) - \beta + \lambda}.
\]

(2.7)

It is clear that the function \(\phi(u, v)\) defined by (2.7) satisfies conditions (i) and (ii) of Lemma 2.1 easily. To verify condition (iii), we proceed as follows.
\[
\text{Re} \phi(iu_1, v_1) = \frac{(1 + \alpha) v_1 \{(1 + \beta) h_1 - \beta + \lambda\}}{[(1 + \beta) h_1 - \beta + \lambda]^2 + [(1 + \beta) h_2]^2}
\]
where \(H(z) = h_1 + ih_2\), \(h_1\) and \(h_2\) being the functions of \(x\) and \(y\) and \(\text{Re} H(z) = h_1 > 0\).

By putting \(v_1 = -\frac{1}{2} (1 + u_2^2)\), we obtain
\[
\text{Re} \phi(iu_2, v_1) = \frac{(1 + da)(1 + u_2^2) \{(1 + \beta) h_1 - \beta + \lambda\}}{[(1 + \beta) h_1 - \beta + \lambda]^2 + [(1 + \beta) h_2]^2} \leq 0
\]

Hence, by Lemma 2.1, \(\text{Re} H(z) > 0\) and this implies that \(I_0(f) \in T(a, c; \alpha, \beta)\). This proves our theorem.

**Corollary 2.1.** Let \(f \in T(a, c; \alpha, \beta)\). Then, for \(\lambda \geq \alpha, \beta > -1\), \(L(a, c) I_0(f) \in K\)

**Proof:** From Theorem 2.2, we clearly see that \(L(a, c) I_0(f) \in K\). The second assertion follows easily from the fact that
\[
L(a, c) I_0(f) = I_0(L(a, c) f(z))
\]

Next we have:

**Theorem 2.3.** Let \(f \in T^*(a, c; \alpha, \beta)\). Then for \(\lambda \geq \alpha, \beta > -1\), \(I_0(f)\) also belongs to \(T^*(a, c; \alpha, \beta)\).

**Proof:** Since
\[
f \in T^*(a, c; \alpha, \beta) \iff zf' \in T(a, c; \alpha, \beta),
\]
we observe, using Theorem 2.2, that
\[
I_0(zf') \in T(a, c; \alpha, \beta).
\]
and this implies that
\[ z(I, f) \in T^*(a, c; \alpha, \beta). \]

Hence \( I, (f) \in T^*(a, c; \alpha, \beta) \). This completes the proof.

**Corollary 2.2.** Let \( f \in T^*(a, c; \alpha, \beta) \). Then, for \( \lambda \geq \alpha, \beta > -1 \). \( I, (L(a, c)f) \in C^* \) and \( I, (L(a, c)f(z)) \in C^* \).

**REFERENCES**

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