On minimal-$\alpha$-spaces

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Abstract. An $\alpha$-space is a topological space in which the topology is generated by the family of all $\alpha$-sets (see [N]). In this paper, minimal-$\alpha\mathcal{P}$-spaces (where $\mathcal{P}$ denotes several separation axioms) are investigated. Some new characterizations of $\alpha$-spaces are also obtained.

Keywords: $\alpha$-space, $\alpha T_i$-space, minimal-$\alpha T_i$ space, $T_2$-closed space, minimal-$T_2$ space, $\psi$-space

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1. Introduction

The family of all topologies on a set $X$ is a complete atomic lattice. There has been a considerable amount of interest in topologies which are minimal or maximal in this lattice with respect to certain topological properties.

Given a topological property $\mathcal{P}$, we say that a topology on a set $X$ is $\mathcal{P}$-minimal if every weaker topology on $X$ does not possess property $\mathcal{P}$.

Throughout this paper, the word “space” will mean topological space, the topology on a space $X$ is denoted by $\tau(X)$, $\text{int}_\tau$ and $\text{cl}_\tau$ (or $\text{int}_X$ and $\text{cl}_X$ when no confusion is possible about the topology on $X$) will denote respectively the interior and the closure operators with respect to $\tau(X)$ and if $\sigma$ is a topology on the underlying set of $X$, then $\sigma$ is called an expansion (respectively a compression) of $\tau(X)$ if $\tau(X) \subseteq \sigma$ (resp. $\sigma \subseteq \tau(X)$).

A subset $R$ of a space $X$ is called regular open if $\text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) = R$. The family of all regular open sets of $X$ is denoted by $RO(X)$ and forms a base for a topology $\tau_s(X)$ on $X$ which is a compression of $\tau(X)$ and it is called the semiregularization of $X$. We say that $X$ is semiregular if $\tau_s(X) = \tau(X)$.

The notion of an $\alpha$-set was introduced in 1965 by Njåstad [N]. Given a space $X$, we say that $A \subseteq X$ is an $\alpha$-set if $A \subseteq \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))$. It is easy to prove that the family $\alpha\tau(X)$ of all $\alpha$-sets of $X$ is a topology on $X$ which will be called the $\alpha$-topology induced by $\tau(X)$ and it consists of all the subsets $A$ of $X$ such that there exists some open set $U \in \tau(X)$ such that $U \subseteq A \subseteq \text{int}_\tau(\text{cl}_\tau(U))$. The members of $\alpha\tau(X)$ will be called the $\alpha$-open sets of $X$ while their complements

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will be called the \( \alpha \)-closed sets of \( X \). Evidently, every open set is an \( \alpha \)-set and hence the \( \alpha \)-topology is an expansion of \( \tau(X) \). We say that \( X \) is an \( \alpha \)-space if \( \tau(X) = \tau_\alpha(X) \).

Obviously, the partial ordered set of all \( \alpha \)-topologies contain both a maximum (the discrete topology) and a minimum (the trivial topology) element. If \( P \) is a topological property, we denote by \( \alpha P \) the class of \( \alpha \)-spaces which satisfy property \( P \). In this paper, we investigate the minimal-\( \alpha P \) property with particular regard to the properties \( P = T_0, T_1, T_2 \) and \( T_{2\frac{1}{2}} \).

### 2. Basic facts on \( \alpha \)-spaces

The \( \alpha \)-topology \( \alpha \tau(X) \) of a space \( (X, \tau(X)) \) has some interesting similarities with the notion of semiregularization \( \tau_s(X) \) (see, for example, [PW]).

**Lemma 1** ([N]). For any \( \alpha \)-open set \( A \) and any \( \alpha \)-closed set \( C \) of a space \( X \), we have:

1. \( \text{cl}_{\alpha \tau}(A) = \text{cl}_\tau(A) \);
2. \( \text{int}_{\alpha \tau}(C) = \text{int}_\tau(C) \);
3. \( \text{int}_{\alpha \tau}(\text{cl}_{\alpha \tau}(A)) = \text{int}_\tau(\text{cl}_\tau(A)) \).

**Proposition 2** ([N]). For any space \( X \), we have \( \alpha(\alpha \tau(X)) = \alpha \tau(X) \).

**Proof:** Evidently \( \alpha \tau(X) \subseteq \alpha(\alpha \tau(X)) \). Let \( B \in \alpha(\alpha \tau(X)) \). Then, there is some \( A \in \alpha \tau(X) \) such that \( A \subseteq B \subseteq \text{int}_{\alpha \tau}(\text{cl}_{\alpha \tau}(A)) \). So, there exists some \( U \in \tau(X) \) such that \( U \subseteq A \subseteq \text{int}_\tau(\text{cl}_\tau(U)) \). Hence, \( \text{int}_\tau(\text{cl}_\tau(U)) = \text{int}_\tau(\text{cl}_\tau(A)) \) and, by Lemma 1(2), we have

\[
U \subseteq A \subseteq B \subseteq \text{int}_{\alpha \tau}(\text{cl}_{\alpha \tau}(A)) = \text{int}_\tau(\text{cl}_\tau(A)) = \text{int}_\tau(\text{cl}_\tau(U)).
\]

This proves that \( B \in \alpha \tau(X) \). \( \Box \)

**Definition 1** ([N]). A space \( X \) is called an \( \alpha \)-space if \( \tau(X) = \alpha \tau(X) \) or, equivalently, if \( \tau(X) = \alpha(\sigma(X)) \) for some topology \( \sigma(X) \) on \( X \).

**Proposition 3.** Let \( (X, \tau(X)) \) be a space. The following are equivalent:

1. \( X \) is an \( \alpha \)-space;
2. \( \tau(X) = \{U \cup \{p\} : U \in \tau(X), p \in \text{int}(\text{cl}(U))\} \);
3. \( \{U \cup \{p\} : U \in \tau(X) \text{ is dense in } X \text{ and } p \in X\} \subseteq \tau(X) \).

**Proof:** (1) \( \Rightarrow \) (2) Evidently, \( \tau(X) \subseteq \{U \cup \{p\} : U \in \tau(X), p \in \text{int}_\tau(\text{cl}_\tau(U))\} \) (it suffices to take \( p \in U \)). Conversely, let \( U \in \tau(X) \) and \( p \in \text{int}_\tau(\text{cl}_\tau(U)) \). Then \( U \subseteq U \cup \{p\} \subseteq U \cup \text{int}_\tau(\text{cl}_\tau(U)) = \text{int}_\tau(\text{cl}_\tau(U)) \). Since \( X \) is an \( \alpha \)-space, \( U \cup \{p\} \in \tau(X) \).

(2) \( \Rightarrow \) (3) Obvious because, when \( \text{cl}_\tau(U) = X \), every \( p \in \text{int}_\tau(\text{cl}_\tau(U)) = X \).
Suppose that \( \{U \cup \{p\} : U \in \tau(X) \text{ is dense in } X \text{ and } p \in X \} \subseteq \tau(X) \) and let \( A \in \alpha \tau(X) \), i.e. \( A \subseteq \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) \). The set \( D = \text{int}_\tau(A) \cup \text{int}_\tau(X \setminus \text{int}_\tau(A)) \) is open and dense because

\[
\text{cl}(D) = \text{cl}_\tau(\text{int}_\tau(A)) \cup \text{cl}_\tau(\text{int}_\tau(X \setminus \text{int}_\tau(A))) \\
= \text{cl}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) \cup \text{cl}_\tau(X \setminus \text{cl}_\tau(\text{int}_\tau(A))) \\
= \text{cl}_\tau(\text{cl}_\tau(\text{int}_\tau(A)) \cup (X \setminus \text{cl}_\tau(\text{int}_\tau(A)))) \\
= \text{cl}_\tau(X) \\
= X.
\]

So, for any \( p \in A \), by hypothesis we have that \( D \cup \{p\} \in \tau(X) \).

Now, we consider the open set:

\[
W_p = (D \cup \{p\}) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) \\
= \left(\text{int}_\tau(A) \cup \text{int}_\tau(X \setminus \text{int}_\tau(A)) \cup \{p\}\right) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) \\
= \left(\text{int}_\tau(A) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))\right) \\
\cup \left(\text{int}_\tau(X \setminus \text{int}_\tau(A)) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))\right) \\
\cup \left(\{p\} \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))\right) \\
= \text{int}_\tau(A) \cup \{p\}
\]

as

\[
(\text{int}_\tau(X \setminus \text{int}_\tau(A))) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) \\
= \left(X \setminus \text{cl}_\tau(\text{int}_\tau(A))\right) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) \\
\subseteq \left(X \setminus \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A)))\right) \cap \text{int}_\tau(\text{cl}_\tau(\text{int}_\tau(A))) = \emptyset.
\]

Thus, for every \( p \in A \), \( W_p = \text{int}_\tau(A) \cup \{p\} \in \tau(X) \) and so, also the union \( \bigcup_{p \in A} W_p = \text{int}_\tau(A) \cup A = A \) is an open set of \( X \). This proves that \( \alpha \tau(X) \subseteq \tau(X) \) and hence that \( X \) is an \( \alpha \)-space. \( \square \)

In [N] it is proved the following:

**Proposition 4.** A space \( X \) is an \( \alpha \)-space if and only if all the nowhere dense sets are closed sets.

The following proposition improves the previous one.
Proposition 5. A space \((X, \tau)\) is an \(\alpha\)-space if and only if every nowhere dense set is discrete.

Proof: Suppose that \(X\) is an \(\alpha\)-space and that \(N\) is a nowhere dense subset of \(X\). By Proposition 4, \(N\) is a closed set and so \(X \setminus N\) is an open dense set. For any \(p \in N\), by Proposition 3(3), the set \(U = (X \setminus N) \cup \{p\}\) is open in \(X\). Since \(U \cap N = \{p\}\), it follows that \(N\) is discrete.

Conversely, let \(U\) be an open dense set of \(X\) and \(p \in X\). Then \(X \setminus U\) is a nowhere dense set and by hypothesis, it is discrete. So, if \(p \in X \setminus U\), there exists some \(V \in \tau\) such that \(V \cap (X \setminus U) = \{p\}\). Hence \(U \cup \{p\} = U \cup V \in \tau(X)\). Since the case when \(p \in U\) is trivial, by Proposition 3(3), it is proved that \(X\) is an \(\alpha\)-space. \(\square\)

It is shown in [Lo] that the operator \(\alpha\) is not monotonic, i.e. that, in general, for two topologies \(\tau(X)\) and \(\sigma(X)\) on a set \(X\), \(\tau(X) \subseteq \sigma(X)\) does not imply \(\alpha\tau(X) \subseteq \alpha\sigma(X)\). However, we have the following:

Lemma 6. Let \(\tau(X)\) and \(\sigma(X)\) be topologies on a set \(X\) such that \(\tau(X) \subseteq \sigma(X)\) and \(\tau_s(X) = \sigma_s(X)\). Then \(\alpha\tau(X) \subseteq \alpha\sigma(X)\).

Proof: Let \(A \in \alpha\tau(X)\). Then there exists some \(U \in \tau(X)\) such that \(U \subseteq A \subseteq \text{int}_\tau(\text{cl}_\tau(U))\). So, being

\[
\text{int}_\sigma(\text{cl}_\sigma(U)) = \text{int}_{\sigma_s}(\text{cl}_{\sigma_s}(U)) = \text{int}_{\tau_s}(\text{cl}_{\tau_s}(U)) = \text{int}_\tau(\text{cl}_\tau(U)),
\]

we have \(U \subseteq A \subseteq \text{int}_{\sigma}(\text{cl}_\sigma(U))\) with \(U \in \tau \subseteq \sigma\), that is \(A \in \alpha\sigma(X)\). \(\square\)

Let us recall that a space \((X, \tau(X))\) is called:

- \(T_2\)-closed (resp. \(T_{2\frac{1}{2}}\)-closed) if it is closed in every Hausdorff (resp. \(T_{2\frac{1}{2}}\)) space containing \(X\) as a subspace,
- minimal-\(T_2\) (resp. minimal-\(T_{2\frac{1}{2}}\)) if it is a \(T_2\) (resp. \(T_{2\frac{1}{2}}\)) space and there is no strictly coarser \(T_2\) (resp. \(T_{2\frac{1}{2}}\)) topology on the same set \(X\).

The following properties are well-known and will be used later.

Proposition 7. A Hausdorff space is \(T_2\)-closed if and only if every open ultrafilter on \(X\) is fixed.

Proposition 8. A space \((X, \tau(X))\) is Hausdorff if and only if its semiregularization \((X, \tau_s(X))\) is Hausdorff.

Corollary 9. A space \((X, \tau(X))\) is \(T_2\)-closed if and only if its semiregularization \((X, \tau_s(X))\) is minimal-\(T_2\).

Proposition 10. Every regular closed subspace of a \(T_2\)-closed space is \(T_2\)-closed.
Proposition 11. A Hausdorff space $X$ is minimal-$T_2$ if and only if it is semi-
regular and $T_2$-closed.

Proposition 12. If $\sigma(X)$ and $\tau(X)$ are two topologies on a set $X$ such that
$\tau_s(X) \subseteq \sigma(X) \subseteq \tau(X)$ then $\tau_s(X) = \sigma_s(X)$.

Proof: Let $R$ be a regular open set of $(X, \tau(X))$. Then there exists some $U \in \tau(X)$ such that $R = \text{int}_\tau(\overline{\text{cl}_\tau(U)})$. So, $R \in \tau_s(X) \subseteq \sigma(X)$. Since $\tau_s(X) \subseteq \sigma(X)$ and using a well-known property of the closure of the semiregularization (see
[PW]), we have that
$$\overline{\text{cl}_\sigma(R)} \subseteq \overline{\text{cl}_{\tau_s}(R)} = \overline{\text{cl}_\tau(R)}$$
while, being $\sigma(X) \subseteq \tau(X)$, it follows
$$\text{int}_\sigma(\overline{\text{cl}_\sigma(R)}) \subseteq \text{int}_\tau(\overline{\text{cl}_\tau(R)}) = R.$$Obviously, being $R \in \sigma(X)$, we also have that $R \subseteq \text{int}_\sigma(\overline{\text{cl}_\sigma(R)})$ and hence that $R = \text{int}_\sigma(\overline{\text{cl}_\sigma(R)})$. Since $R \in \sigma(X)$, this means that $R$ is a regular open set of $(X, \sigma(X))$. On the other hand, let $S$ be a regular open set of $(X, \sigma(X))$. Then there exists some $V \in \sigma(X)$ such that $S = \text{int}_\sigma(\overline{\text{cl}_\sigma(V)})$. Since $S \in \sigma(X) \subseteq \tau(X)$, we have that
$$\overline{\text{cl}_\tau(S)} \subseteq \overline{\text{cl}_\sigma(S)}$$Hence, being $\tau_s(X) \subseteq \sigma(X)$ and by some well-known properties of the interior of
the semiregularization it follows that
$$\text{int}_\tau(\overline{\text{cl}_\tau(S)}) = \text{int}_{\tau_s}(\overline{\text{cl}_{\tau_s}(S)}) \subseteq \text{int}_\sigma(\overline{\text{cl}_\sigma(S)}) = S.$$Obviously, being $S \in \tau(X)$, we also have that $S \subseteq \text{int}_\tau(\overline{\text{cl}_\tau(S)})$ and hence that $S = \text{int}_\tau(\overline{\text{cl}_\tau(S)})$, that is $S$ is a regular open set of $(X, \tau(X))$. Thus, the topolo-
gies generated by these families of regular open sets, i.e. the semiregularization of
$(X, \tau(X))$ and $(X, \sigma(X))$ coincide and we can conclude that $\tau_s(X) = \sigma_s(X)$. □

Proposition 13. Let $\mathcal{U}$ be a free open ultrafilter on a Hausdorff space $X$ and $p$
be a fixed point in $X$. Then, there exists a Hausdorff topology $\tau_\mathcal{U}$ on $X$ such that
$\alpha_\mathcal{U}(X) \subseteq \alpha \tau(X)$.

Proof: Let us consider the family $\tau_\mathcal{U}(X) = \{U \in \tau(X) : p \in U \Rightarrow U \in \mathcal{U}\}$. It is a simple routine to verify that $\tau_\mathcal{U}(X)$ forms a topology on $X$ such that $\tau_\mathcal{U}(X) \subseteq \tau(X)$.

The space $(X, \tau_\mathcal{U}(X))$ is $T_2$. In fact, for every $x \neq y$ in $X$, since $(X, \tau(X))$
is Hausdorff, there are $U, V \in \tau(X)$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. If
$p \notin U \cup V$, $U, V \in \tau_\mathcal{U}(X)$ and we are done. Otherwise, if, for example, $p \in U$, we
have $p \notin V$ and $V \in \tau(X)$. Furthermore, since $\mathcal{U}$ is free with respect to $(X, \tau(X))$,
there exist some $N \in \tau(X)$ and some $W \in \mathcal{U}$ such that $y \in N$ and $W \cap N = \emptyset$. 

On minimal-α-spaces
Thus $W \subseteq U \cup W$ implies $U \cup W \in \mathcal{U}$ and hence $U \cup W \in \tau_\mathcal{U}(X)$, while being $p \notin V \cap N \in \tau(X)$ it follows that $V \cap N \in \tau_\mathcal{U}(X)$. So, $U \cup W$ and $V \cap N$ are two open neighborhoods of $x$ and $y$ respectively in $\tau_\mathcal{U}(X)$ such that:

$$(U \cup W) \cap (V \cap N) = (U \cup (V \cap N)) \cup (W \cap (V \cap N))$$

$$\subseteq (U \cup V) \cup (W \cap N)$$

$$= \emptyset \cup \emptyset = \emptyset$$

and this proves that the space $(X, \tau_\mathcal{U}(X))$ is Hausdorff.

Since it is immediate to see that every neighborhood of $p$ in $\tau_\mathcal{U}(X)$ belongs to $\mathcal{U}$ and $(X, \tau_\mathcal{U}(X))$ is $T_2$, it follows that $p$ is the unique convergence point of $\mathcal{U}$ with respect to $\tau_\mathcal{U}(X)$.

In order to show that $\alpha\tau_\mathcal{U}(X) \subseteq \alpha\tau(X)$, we observe first that, for every $U \in \tau_\mathcal{U}(X)$, it results:

$$(1) \quad \text{cl}_{\tau_\mathcal{U}}(U) = \begin{cases} \text{cl}_\tau(U) & \text{if } U \notin \mathcal{U}, \\ \text{cl}_\tau(U) \cup \{p\} & \text{if } U \in \mathcal{U}. \end{cases}$$

In fact, $\tau(X) \subseteq \tau_\mathcal{U}(X)$ implies, in any case, $\text{cl}_{\tau_\mathcal{U}}(U) \subseteq \text{cl}_\tau(U)$.

Now, consider the case $U \notin \mathcal{U}$ and suppose, by contradiction that there is some $x \in \text{cl}_{\tau_\mathcal{U}}(U)$ such that $x \notin \text{cl}_\tau(U)$. So, there exists some neighborhood $N$ of $x$ in $\tau(X)$ such that $N \cap U = \emptyset$.

Since $\mathcal{U}$ is an open ultrafilter on $(X, \tau(X))$, $U \notin \mathcal{U}$ implies that $X \setminus \text{cl}_\tau(U) \in \mathcal{U}$. Now, $W = (X \setminus \text{cl}_\tau(U)) \cup N$ is an open neighborhood of $x$ with respect to $\tau_\mathcal{U}(X)$ (because $W \in \tau(X)$, $x \in N \subseteq W$, and $X \setminus \text{cl}_\tau(U) \subseteq W$ implies $W \in \mathcal{U}$) but it results

$$W \cap U = ((X \setminus \text{cl}_\tau(U)) \cup N) \cap U = ((X \setminus \text{cl}_\tau(U)) \cap U) \cup (N \cap U) = \emptyset \cup \emptyset = \emptyset$$

which is a contradiction to $x \in \text{cl}_{\tau_\mathcal{U}}(U)$.

Let us consider the case $U \in \mathcal{U}$. Evidently $p \in \text{cl}_{\tau_\mathcal{U}}(U)$ as for every open neighborhood $N$ of $p$ in $\tau_\mathcal{U}(X)$, it follows that $N \in \mathcal{U}$ and hence $N \cap U \in \mathcal{U}$ implies $N \cap V \neq \emptyset$. Conversely, suppose, by contradiction, that there is some $x \in \text{cl}_{\tau_\mathcal{U}}(U)$ such that $x \neq p$ and $x \notin \text{cl}_\tau(U)$. Then, there exists some open neighborhood $N$ of $p$ in $\tau(X)$ such that $N \cap U = \emptyset$ and, it must be $N \notin \tau_\mathcal{U}(X)$, that is $p \in N \notin \mathcal{U}$. Since $(X, \tau(X))$ is $T_2$ and $x \neq p$, there is some open neighborhood $G$ of $x \in \tau(X)$ such that $p \notin G$. Thus, $p \notin N \cap G \in \tau(X)$ implies that $N \cap G \in \tau_\mathcal{U}(X)$. So, $N \cap G$ is an open neighborhood of $x$ in $\tau_\mathcal{U}(X)$ such that $(N \cap G) \cap U \subseteq N \cap U = \emptyset$. A contradiction to $x \in \text{cl}_{\tau_\mathcal{U}}(U)$. Applying the usual duality rules to formula (1), we also obtain that, for every $U \in \tau_\mathcal{U}(X)$, it results:

$$(2) \quad \text{int}_{\tau_\mathcal{U}}(\text{cl}_{\tau_\mathcal{U}}(U)) = \begin{cases} \text{int}_\tau(\text{cl}_\tau(U)) \setminus \{p\} & \text{if } U \notin \mathcal{U}, \\ \text{int}_\tau(\text{cl}_\tau(U)) & \text{if } U \in \mathcal{U}. \end{cases}$$
In fact, if \( U \notin \mathcal{U} \), as \( \mathcal{U} \) is an open ultrafilter on \((X, \tau(X))\), \( X \setminus \text{cl}_\tau(U) \in \mathcal{U} \) and so we have that:

\[
\text{int}_{\tau(\mathcal{U})} \left( \text{cl}_{\tau, \mathcal{U}}(U) \right) = \text{int}_{\tau(\mathcal{U})} \left( \text{cl}_\tau(U) \right)
\]

\[
= X \setminus \text{cl}_{\tau(\mathcal{U})} \left( X \setminus (\text{cl}_\tau(U)) \right)
\]

\[
= X \setminus \text{cl}_\tau \left( X \setminus (\text{cl}_\tau(U) \cup \{p\}) \right)
\]

\[
= \left( X \setminus \text{cl}_\tau \left( X \setminus \text{cl}_\tau(U) \right) \right) \cap (X \setminus \{p\})
\]

\[
= \text{int}_\tau \left( \text{cl}_\tau(U) \right) \setminus \{p\}.
\]

If \( U \in \mathcal{U} \), since \( \mathcal{U} \) is an open ultrafilter on \((X, \tau(X))\), it necessarily results \( X \setminus \text{cl}_\tau(U) \notin \mathcal{U} \) and we have:

\[
\text{int}_{\tau(\mathcal{U})} \left( \text{cl}_{\tau(\mathcal{U})}(U) \right) = X \setminus \text{cl}_{\tau(\mathcal{U})} \left( X \setminus \text{cl}_{\tau(\mathcal{U})}(U) \right)
\]

\[
= X \setminus \text{cl}_\tau \left( X \setminus \text{cl}_{\tau(\mathcal{U})}(U) \right)
\]

\[
= \text{int}_\tau \left( \text{cl}_{\tau(\mathcal{U})}(U) \right)
\]

\[
= \text{int}_\tau \left( \text{cl}_\tau(U) \cup \{p\} \right)
\]

\[
= \text{int}_\tau \left( \text{cl}_\tau(U) \right),
\]

where the last equality is due to the fact that \( p \in \text{cl}_\tau(U) \) (because for every neighborhood \( N \) of \( p \) in \( \tau(X) \), we have \( N \in \mathcal{U} \) and so, being \( U \in \mathcal{U} \), it follows that \( N \cap U \in \mathcal{U} \) and \( N \cap U \neq \emptyset \)).

Now, for every \( A \in \alpha_{\tau(\mathcal{U})}(X) \), there exists some \( U \in \tau(\mathcal{U})(X) \) such that \( U \subseteq A \subseteq \text{int}_{\tau(\mathcal{U})} \left( \text{cl}_{\tau(\mathcal{U})}(U) \right) \) and by formula (2), it immediately follows, in both cases, that \( U \subseteq A \subseteq \text{int}_\tau \left( \text{cl}_\tau(U) \right) \) with \( U \in \alpha_\tau(X) \subseteq \tau(X) \) and so that \( A \in \alpha_\tau(X) \). Thus \( \alpha_{\tau(\mathcal{U})}(X) \subseteq \alpha_\tau(X) \). To finish the proof, we will show that \( \alpha_{\tau(\mathcal{U})}(X) = \alpha_\tau(X) \). Since \( \mathcal{U} \) is free, \( p \) is not an adherent point for \( \mathcal{U} \) and so there exist some neighborhood \( V \) of \( p \) in \( \tau(X) \) and some \( U \in \mathcal{U} \) such that \( U \cap V = \emptyset \). Thus \( V \notin \mathcal{U} \) and, being \( p \in V \), by definition of \( \tau(\mathcal{U})(X) \), it follows that \( V \notin \tau(\mathcal{U})(X) \). Evidently \( V \in \alpha_\tau(X) \) (as \( V \in \tau(X) \)) but \( V \notin \alpha_{\tau(\mathcal{U})}(X) \). Suppose, by contradiction, that there exists some \( W \in \tau(\mathcal{U})(X) \) such that \( W \subseteq V \subseteq \text{int}_{\tau(\mathcal{U})} \left( \text{cl}_{\tau(\mathcal{U})}(W) \right) \). Since \( V \notin \mathcal{U} \), it follows a fortiori that \( W \notin \mathcal{U} \) and so, by formula (2), we have that \( W \subseteq V \subseteq \text{int}_\tau \left( \text{cl}_\tau(W) \right) \setminus \{p\} \) and thus that \( p \notin V \). A contradiction. This proves that \( \alpha_{\tau(\mathcal{U})}(X) \subseteq \alpha_\tau(X) \) and concludes our proof. \( \square \)

3. Minimal-\( \alpha \)-\( T_0 \)-spaces

**Definition 2.** Let \( \mathcal{P} \) be a topological property. A space \( X \) is said to be an \( \alpha \mathcal{P} \)-space if it is an \( \alpha \)-space and property \( \mathcal{P} \) holds.
**Definition 3.** Let $\mathcal{P}$ be a topological property. A space $X$ is called *minimal-$\alpha\mathcal{P}$* if it is an $\alpha\mathcal{P}$-space and there is no strictly coarser $\alpha\mathcal{P}$-topology on the same set.

Since the unique $T_1$-minimal-space is the cofinite topology which is an $\alpha$-space (by Proposition 3), it is evident that the class of minimal-$\alpha T_1$-spaces coincides with the well-known class of minimal-$T_1$.

One would suspect that if $X$ is minimal-$T_0$, then $(X, \alpha\tau(X))$ is minimal-$\alpha T_0$. Here is a counterexample.

**Example 14.** Consider $\mathbb{R}$ with $\tau(\mathbb{R}) = \{(-\infty, a) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$. Then $\mathbb{R}$ is minimal-$T_0$. If $\emptyset \neq U \subseteq \tau(\mathbb{R})$, $\text{cl}_\mathbb{R}(U) = \mathbb{R}$, then $\text{int}_\mathbb{R}(\text{cl}_\mathbb{R}(U)) = \mathbb{R}$. So, if $a < b$ and $c = a - 1$, then $(-\infty, c) \cup \{a\}, (-\infty, c) \cup \{b\} \in \alpha\tau(\mathbb{R})$. In particular, it follows that $(\mathbb{R}, \alpha\tau(\mathbb{R}))$ is $T_1$. If $\sigma$ is the cofinite topology on $\mathbb{R}$, then $\sigma \subset \tau(\mathbb{R})$.

**Lemma 15.** Let $X$ be a set, $p \in X$, and $\tau(X) = \{U : p \in U \text{ and } X \setminus U \text{ is finite}\} \cup \{\emptyset\}$. Then $X$ is minimal-$\alpha T_0$.

**Proof:** Clearly, $X$ is $T_0$. If $\emptyset \neq U \subseteq \tau(X)$, then $p \in U$, $\text{cl}_X U = X$, and $\text{int}_X \text{cl}_X U = X$. Now $U \cup \{q\}$ is open for all $q \in X$, by Proposition 3, since $X$ is $\alpha T_0$. Let $\sigma \subseteq \tau(X)$ and $(X, \sigma)$ be $\alpha T_0$. If $\emptyset \neq U \subseteq \tau(X)$, then $p \in U$, and $X \setminus U = \{q_1, \ldots, q_n\}$ is finite. Since $\emptyset \neq V \subseteq \tau(X)$ implies $p \in V$, then for each $1 \leq i \leq n$, there is a $V_i \in \sigma$ such that $p \in V_i \subseteq X \setminus \{q_i\}$. For $T = \bigcap\{V_i : 1 \leq i \leq n\}, p \in T \in \sigma$ and $T \subseteq U$. As $(X, \sigma)$ is $\alpha T_0$ and $\text{cl}_\sigma T = X$, then $U = \bigcup\{T \cup \{q\} : q \in U \setminus T\} \in \sigma$. \qed

In the topology just considered, every point different from $p$ is closed. Now, let us consider the case in which some point of $X$ is not closed.

**Lemma 16.** Let $X$ be an $\alpha T_0$-space, and $p \in X$ such that $\text{cl}_X \{p\} \neq \{p\}$. Then $\text{cl}_X(\{p\})$ is a regular-closed set and $p \in \text{int}_X(\text{cl}_X(\{p\}))$.

**Proof:** Let $U = X \setminus \text{cl}_X(\{p\})$. It suffices to show that $U$ is regular-open. If $p \in \text{cl}_X(U)$, then $\text{cl}_X(U) = X$ and $\text{int}_X(\text{cl}_X(U)) = X$. So, for $q \in \text{cl}_X(\{p\}) \setminus \{p\}$, $U \cup \{q\}$ is open and $(U \cup \{q\}) \cap \{p\} = \emptyset$. That is, $q \notin \text{cl}_X(\{p\})$, a contradiction. So, $p \notin \text{cl}_X(U)$. For $V = X \setminus \text{cl}_X(U)$, we have that $p \in V \subseteq \text{cl}_X(\{p\})$. Thus, $\text{cl}_X(V) = \text{cl}_X(\{p\})$. \qed

In order to obtain a characterization of minimal-$T_0$-spaces, we need some other lemmas.

**Lemma 17.** Let $X$ be an $\alpha T_0$-space, and $p, q \in X$ such that $\text{cl}_X(\{p\}) \neq \{p\}, \text{cl}_X(\{q\}) \neq \{q\}$, and $p \neq q$. Then $p \notin \text{cl}_X(\{q\})$ and $q \notin \text{cl}_X(\{p\})$.

**Proof:** Since $X$ is $T_0$, then $\text{cl}_X(\{p\}) \neq \text{cl}_X(\{q\})$. Assume that $q \in \text{cl}_X(\{p\})$. Then $p \notin \text{cl}_X(\{q\})$ as $\text{cl}_X(\{p\}) \neq \text{cl}_X(\{q\})$. So, $\text{cl}_X(\{q\}) \subseteq \text{cl}_X(\{p\})$. By Lemma 16, $q \in \text{int}_X(\text{cl}_X(\{q\})) \subseteq \text{int}_X(\text{cl}_X(\{p\})) \subseteq \text{cl}_X(\{p\})$. As $p \notin \text{cl}_X(\{q\})$, it follows that $p \notin \text{int}_X(\text{cl}_X(\{q\}))$. Therefore, $q \notin \text{cl}_X(\{p\})$, a contradiction. \qed
Lemma 18. Let $X$ be a minimal-$\alpha T_0$-space, and $p, q \in X$ such that $\text{cl}_X\{p\} \neq \{p\}$, $\text{cl}_X\{q\} \neq \{q\}$, and $p \neq q$. Then $\text{int}_X \left( \text{cl}_X\{p\} \right) = \{p\}$ and $\text{int}_X \left( \text{cl}_X\{q\} \right) = \{q\}$.

Proof: Let $\tau = \tau(X)$ be the topology on $X$, $r \in \text{int}_X \left( \text{cl}_X\{p\} \right) \setminus \{p\}$ and consider the topology $\sigma = \{U \in \tau : r \in U \text{ implies } q \in U\}$ on $X$. Clearly, $\sigma \subseteq \tau$. There are a number of cases to verify that $(X, \sigma)$ is $T_0$ but each case is straightforward. Next we show that $(X, \sigma)$ is an $\alpha$-space. Let $U$ be open and dense in $(X, \sigma)$ and $t \in X \setminus U$. We want to show that $U \cup \{t\} \in \sigma$. If $r \in U$, then $U$ is dense and open in $(X, \tau)$ and hence, $U \cup \{t\} \in \sigma$. Suppose that $r \notin U$. Our first goal is to show that $U$ is dense in $\tau$. If $\text{cl}_\tau(U) \neq \text{cl}_\sigma(U)$, then $r \in \text{cl}_\sigma(U) \setminus \text{cl}_\tau(U)$. Then $p \notin U$ and $\text{cl}_\tau\{p\} \cap U = \emptyset$. In particular, $\text{int}_\tau\{p\} \cap U = \emptyset$. There is an $V \in \tau$ such that $p \in V$ and $r \notin V$. Now $p \in \text{int}_\tau\{p\} \cap V \in \sigma$ and $\text{int}_\tau\{p\} \cap V \cap U = \emptyset$. So, $U$ is not dense in $(X, \sigma)$, a contradiction. Thus, $U$ is dense in $\tau$. Also the above proof shows that $p \in U$. If $q \notin U$, then $\text{cl}_\tau\{q\} \cap U = \emptyset$. In particular, $\text{int}_\tau\{q\} \cap U = \emptyset$. But $q \in \text{int}_\tau\{q\} \in \sigma$, a contradiction. Thus, $q \in U$. With both $p, q \in U$, we have that $U \cup \{r\} \in \sigma$. For $t \neq r$, then $U \cup \{t\} \in \sigma$ as $U \cup \{t\} \in \tau$ and $r \notin U \cup \{t\}$. □

Proposition 19. Let $X$ be a minimal-$\alpha T_0$-space, and $P = \{p \in X : \text{cl}_X\{p\} \neq \{p\}\}$ such that $|P| \geq 2$. Then $P$ is dense in $X$ and if $V \in \tau(X)$ and $V \setminus P \neq \emptyset$, then $P \subseteq V$.

Proof: Let $\tau = \tau(X)$ be the topology of the space $X$. First we show that $Q = \{q \in X : \{q\} \in \tau\}$ is dense. Clearly, $Q \supseteq P$. Fix $r \in X \setminus \text{cl}_X(Q)$. Note that the topology $\sigma = \{U \in \tau : r \in U \text{ implies } Q \subseteq U\} \subseteq \tau$. If we show that $(X, \sigma)$ is $\alpha T_0$, it will follow that $Q$ is dense. Since $\{q\} \in \sigma$ for all $q \in Q$, $q$ can be $T_0$-separated from all $p \in X \setminus \{q\}$ in $\sigma$. As $(X \setminus \text{cl}_X(Q)) \cup Q \in \sigma$ and $(X \setminus \text{cl}_X(Q)) \cup Q \cup \{t\} \in \sigma$ for $t \in \text{cl}_X(Q) \setminus Q$, a point $t \in \text{cl}_X(Q) \setminus Q$ can be $T_0$-separated from all $p \in X \setminus \{t\}$ in $\sigma$. Let $s, t \in X \setminus \text{cl}_X(Q)$ As $\tau$ is $T_0$, there is some $V \in \tau$ such that $s \in V$ and $t \notin V$ or vice versa. Now, $s \in V \cup Q \in \sigma$ and $t \notin V \cup Q$. This completes the proof that $(X, \sigma)$ is $T_0$. Next we show that $(X, \sigma)$ is an $\alpha$-space. Let $U$ be an open and dense subset of $(X, \sigma)$. Note that $Q \subseteq U$. If $\text{cl}_\tau(U) \neq \text{cl}_\sigma(U)$, then $r \in \text{cl}_\sigma(U) \setminus \text{cl}_\tau(U)$. There is an $V \in \tau$ such that $r \in V$ and $V \cap Q = \emptyset$. As $r \notin P$, $\text{cl}_\{r\} = \{r\}$, $\emptyset \neq V \setminus \{r\} \in \sigma$ and $V \setminus \{r\} \cap U = \emptyset$, a contradiction as $U$ is dense in $(X, \sigma)$. This shows that $\text{cl}_\tau(U) = X$. As $\tau$ is an $\alpha$-topology, for $t \in X, U \cup \{t\} \in \tau$. As $Q \subseteq U$, $U \cup \{t\} \in \sigma$. This completes the proof that $(X, \sigma)$ is an $\alpha T_0$-space. Finally, we show that $Q = P$. Assume that $q \in Q \setminus P$. The topology $\sigma = \{U \in \tau : q \in U \text{ implies } P \subseteq U\} \subseteq \tau$. Similar to the above, it is straightforward to show that $(X, \sigma)$ is $T_0$ and an $\alpha$-space, a contradiction as $\tau$ is minimal-$\alpha T_0$. Thus, $P = Q$. □
Proposition 20. Let $X$ be a minimal-$\alpha T_0$-space, and $P = \{p \in X : \text{cl}_X(\{p\}) \neq \{p\}\}$ such that $|P| \geq 2$. Then $\tau(X)$ is generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \cup \{\{p\} : p \in P\}$.

Proof: This is an obvious consequence of Proposition 19. \qed

Proposition 21. Let $X$ be a set and $P \subseteq X$ such that $|P| \geq 2$. Let $\tau(X)$ be generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \cup \{\{p\} : p \in P\}$. Then $X$ is a minimal-$\alpha T_0$-space.

Proof: Let $\tau = \tau(X)$ be the topology on $X$ and $\sigma \subseteq \tau$ be an $\alpha T_0$ topology. For $p \in P, (X \setminus P) \cup \{p\} = \text{cl}_\tau(\{p\}) \subseteq \text{cl}_\sigma(\{p\})$. Thus, for all $p \in P$, $\text{cl}_\sigma(\{p\}) \neq \{p\}$.

By Lemma 17, for $q \in P \setminus \{p\}$, $q \notin \text{cl}_\sigma(\{p\})$. Hence $\text{cl}_\sigma(\{p\}) = (X \setminus P) \cup \{p\} = \text{cl}_\tau(\{p\})$. Thus, $P \setminus \{p\} \in \sigma$ for all $p \in P$. By Lemma 16, $\sigma \subseteq \text{int}_\tau(\text{cl}_\sigma(\{p\}))$. As $|P| \geq 2$, let $q \in P \setminus \{p\}$. Then $\text{int}_\sigma(\text{cl}_\sigma(\{p\})) \cap P \setminus \{p\} = \{q\} \in \sigma$ and $P \in \sigma$. As $X = \text{cl}_\tau(P) \subseteq \text{cl}_\sigma(P)$, $X = \text{int}_\sigma(\text{cl}_\sigma(P))$. Since $(X, \sigma)$ is an $\alpha T_0$-space, it follows that for $q \in X \setminus P, P \cup \{q\} \in \sigma$. This shows that $\sigma = \tau$. \qed

Finally, we obtain the characterization of $\alpha T_0$-spaces.

Theorem 22. Let $X$ be an $\alpha T_0$-space and $P = \{p \in X : \text{cl}_X(\{p\}) \neq \{p\}\}$. Then $X$ is minimal-$\alpha T_0$ iff $P \neq \emptyset$ and

(i) if $P = \{p\}$, then $\tau(X) = \{U : p \in U$ and $X \setminus U$ is finite$\} \cup \{\emptyset\}$, or

(ii) if $|P| \geq 2$, then $\tau(X)$ is generated by the base $\{P \cup \{q\} : q \in X \setminus P\} \cup \{\{p\} : p \in P\}$.

Proof: ($\Leftarrow$) It follows from Lemma 18 and Proposition 20. ($\Rightarrow$) It follows from Lemma 15 and Proposition 21. \qed

4. Minimal-$\alpha T_2$-spaces

Proposition 23. Let $X$ be an $\alpha T_2$-space. Then $X$ is a minimal-$\alpha T_2$-space if and only if $(X, \tau_s(X))$ is minimal-$T_2$ and $\tau(X) = \alpha \tau_s(X)$.

Proof: ($\Rightarrow$) Suppose that $X$ is a minimal-$\alpha T_2$-space. Thus $\alpha \tau(X) = \tau(X)$. Then the space $(X, \tau(X))$ is $T_2$-closed. In fact, if, by contradiction, it is not, by Proposition 7, there exists some free open ultrafilter $U$ on $(X, \tau(X))$ and by Proposition 13, there exists a strictly coarser $\alpha T_2$ topology $\tau_U$. A contradiction to the $\alpha$-minimality of $(X, \tau(X))$. Hence, by Corollary 9, $(X, \tau_s(X))$ is minimal-$T_2$. Furthermore, being $\tau_s(X) \subseteq \tau(X)$ and, obviously, $(\tau_s(X))_s = \tau_s(X)$, by Lemma 6, we have that $\alpha \tau_s(X) \subseteq \alpha \tau(X)$, i.e. $\alpha \tau_s(X) \subseteq \tau(X)$ where $\alpha \tau_s(X)$ is Hausdorff by Proposition 11 as $\tau(X)$ is Hausdorff and the $T_2$ axioms are expansive. Since $(X, \tau(X))$ is minimal-$\alpha T_2$, we conclude that $\alpha \tau_s(X) = \tau(X)$.

($\Leftarrow$) Let us suppose that $(X, \tau_s(X))$ is minimal-$T_2$ and $\tau(X) = \alpha \tau_s(X)$. Then $(X, \tau(X))$ is $H$-closed by Corollary 9. Now, let $\sigma(X)$ be an $\alpha T_2$ topology on $X$ such that

$\sigma(X) \subseteq \tau(X)$. 
For every $R$ regular open set of $(X, \tau(X))$, $X \setminus R$ is a regular closed set and hence, by Proposition 10, it is a $T_2$-closed subspace of the Hausdorff space $(X, \sigma(X))$. Thus $X \setminus R$ is a closed set of $(X, \sigma(X))$ and $R \in \sigma(X)$. This proves that

$$\tau_s(X) \subseteq \sigma(X).$$

Hence, by Proposition 12, $\tau_s(X) = \sigma_s(X)$ and so, applying Lemma 6 to $\tau_s(X) \subseteq \sigma(X)$, we have that $\alpha \tau_s(X) \subseteq \alpha \sigma(X)$. Since $\sigma(X)$ is an $\alpha$ topology and, by hypothesis $\alpha \tau_s(X) = \tau(X)$, it follows that $\tau(X) \subseteq \sigma(X)$ and so that $\tau(X) = \sigma(X)$. This shows that $(X, \tau(X))$ is a minimal-$\alpha T_2$-space and concludes our proof. □

**Corollary 24.** Let $(X, \tau(X))$ be a space with a dense set $D$ of isolated points. Then:

1. the $\alpha$-topology $\alpha \tau(X)$ coincides with the topology generated by $\tau(X) \cup \{D \cup \{x\} : x \in X\}$, i.e. it is a simple extension of the subspace $D$;
2. $X$ is an $\alpha$-space if and only if $X \setminus D$ is discrete;
3. if $X$ is a semiregular $\alpha$-space, it results:

$$\{\sigma : \sigma \text{ is a topology on } X \text{ such that } \sigma_s = \tau(X)\} = \{\tau(X)\}.$$

**Proof:** Straightforward applications of Propositions 3, 5 and 6. □

**Example 25.** Let us consider the set $X = \mathbb{R} \times [0, +\infty[$. It is easy to verify that

$$\tau_s(X) = \{U \subseteq X : (x,0) \in U \Rightarrow \exists \epsilon > 0 \text{ such that } |x - \epsilon, x + \epsilon| \subseteq U\}$$

defines a Tychonoff topology on $X$ and that $D = \mathbb{R} \times ]0, +\infty[$ is a dense set of isolated points. Now, if we consider another topology on $X$,

$$\tau(X) = \{U \subseteq X : (x,0) \in U \setminus Q \times \{0\} \Rightarrow \exists \epsilon > 0 \text{ such that } |x - \epsilon, x + \epsilon| \subseteq U\}$$

such that $(x,0) \in U \cap Q \times \{0\} \Rightarrow \exists \epsilon > 0$ such that

$$|x - \epsilon, x + \epsilon| \subseteq U \cup (x - \epsilon, x + \epsilon) \cap Q \times \{0\} \subseteq U,$$

it is a simple routine to check that $\tau_s(X)$ is the semiregularization of $\tau(X)$ and that $\tau(X) \neq \tau_s(X)$. Furthermore, since $C = X \setminus (\mathbb{R} \times ]0, +\infty[)$ is a closed nowhere dense subset with respect to both $\tau(X)$ and $\tau_s(X)$, by Corollary 24 follows that $\alpha \tau(X) = \alpha \tau_s(X)$ coincides with the topology generated by $\tau(X) \cup \{(x,0) : x \in \mathbb{R}\}$ and so that $\tau(X) \neq \alpha \tau(X)$. Thus, we have that

$$\tau_s(X) \subseteq \tau(X) \subseteq \alpha \tau(X) = \alpha \tau_s(X).$$
Example 26 (A Tychonoff $\alpha$-space which is not minimal-$T_2$). Let us recall that two sets are almost disjoint if their intersection is a finite set. It is a simple routine to show (by using Zorn’s Lemma) that there exists a maximal almost disjoint family $M$ of subsets of $\mathbb{N}$. The space generated on the set $\psi = \mathbb{N} \cup M$ by the base

$$B = \{\{n\} : n \in \mathbb{N}\} \cup \{\{M\} \cup S : M \in M \text{ and } S \text{ is a cofinite subset of } M\}$$

is a 0-dimensional (and hence Tychonoff) but not normal (and hence not minimal-$T_2$). This space is known in literature as the $\psi$-space (see 1N, [PW]). Since, every closed nowhere dense set of $\psi$ is discrete, by Proposition 5, it is evident that $\psi$ is an $\alpha$-space.

Example 27 (A minimal-$T_2$, $\alpha$-space). Let us consider the set

$$Z = \left\{\left(\frac{1}{n}, 0\right) : n \in \mathbb{N}\right\} \cup \left\{\left(\frac{1}{n}, \frac{1}{m}\right) : n \in \mathbb{N}, m \in \mathbb{Z}\right\}$$

with the topology $\tau(Z)$ induced by the usual topology on $\mathbb{R}^2$. Let $X = Z \cup \{a, b\}$ and define a topology $\tau(X)$ on $X$ by saying that a subset $U \subset Z$ is open if $U \cap Z \in \tau(Z)$ and if $a \in U$ (respectively, $b \in U$) there exists some $k \in \mathbb{N}$ such that $\{(\frac{1}{n}, \frac{1}{m}) : n \in \mathbb{N}, m \geq k\} \subseteq U$ (respectively, $\{(\frac{1}{n}, \frac{1}{m}) : n \in \mathbb{N}, m \geq k\} \subseteq U$). It is well-known (see 4.8(d), [PW]) that the space $X$ is Urysohn, not compact and minimal-$T_2$. Furthermore, since every its nowhere dense subset (namely, $\{(\frac{1}{n}, 0) : n \in \mathbb{N}\} \cup \{a, b\}$) is discrete, it follows from Proposition 5 that $X$ is an $\alpha$-space.

Let us note that, by 24(3), in both of the above spaces, we have that $\{\sigma : \sigma$ is a topology on $X$ such that $\sigma_s = \tau(X)\} = \{\tau(X)\}$. Thus, the question remains: is there some semiregular $\alpha$-space $(X, \tau(X))$ such that $\{\tau(X)\} \subsetneq \{\sigma : \sigma$ is a topology on $X$ such that $\sigma_s = \tau(X)\}$?

We now provide an example of a Tychonoff $\alpha$-space $X$ such that

$$\{\tau(X)\} \subsetneq \{\sigma : \sigma$ is a topology on $X$ such that $\sigma_s = \tau(X)\}.$$ 

Example 28. Recall that a measurable subset $A$ of $\mathbb{R}$ has density 1 if

$$A = \{x \in \mathbb{R} : \lim_{h \to 0} \frac{m(A \cap [x-h,x+h])}{2h} = 1\}.$$

The set of $\{A \subseteq \mathbb{R} : A$ measurable with density 1\} forms a topology $\delta(\mathbb{R})$, called the density topology, on the set $\mathbb{R}$. The space $(\mathbb{R}, \delta(\mathbb{R}))$ is a Tychonoff space without isolated points, strictly finer than the usual topology $\tau(\mathbb{R})$, and has the property that every nowhere dense subset is closed and discrete (see 2.7 in [T]).
particular, \((\mathbb{R}, \delta(\mathbb{R}))\) is an \(\alpha\)-space by Proposition 4. We need one additional fact about \((\mathbb{R}, \delta(\mathbb{R}))\). Note that for \(x \in U \in \delta(\mathbb{R}), U \cap (x, \infty) \neq \emptyset\) and \(U \cap (-\infty, x) \neq \emptyset\). In particular, the \(\delta(\mathbb{R})\) open neighborhood filter is contained in two distinct \(\delta(\mathbb{R})\) open ultrafilters on \(\mathbb{R}\). Before continuing with this example, we need a result about absolutes of spaces.

For a Hausdorff space \(X\) and let \(EX = \{U : U\) is a convergent, open ultrafilter on \(X\}\). For \(U \in \tau(X)\), let \(O(U) = \{U \in EX : U \in U\}\). For \(U, V \in \tau(X)\), it is easy to verify (see [PW]) that \(O(\emptyset) = \emptyset, O(X) = EX, O(U \cap V) = O(U) \cap O(V), O(U \cup V) = O(U) \cup O(V), EX \setminus O(U) = O(X \setminus \text{cl}X(U))\), and \(O(U) = O(\text{int}_X \text{cl}_X(U))\). EX with the topology generated by \(\{O(U) : U \in \tau(X)\}\) is an extremally disconnected Tychonoff space, called the absolute of \(X\). The function \(k : EX \to X\) defined by \(k(U)\) is the unique convergent point of \(U\) is called a covering function and has the properties that \(k\) is irreducible, \(\theta\)-continuous, perfect and onto. If \(X\) is regular, then \(k\) is also continuous. If \(D \subseteq EX\) such that \(k[D] = X\), then \(D\) is dense in \(EX\).

**Proposition 29.** Let \(X\) be a regular \(\alpha\)-space and \(Y\) a subspace of \(EX\) such that for each \(x \in X\), \(|k^{-1}(\langle x \rangle)| = 2\). Then \(Y\) is an \(\alpha\)-space.

**Proof:** Let \(\overline{k} = k|Y\). The function \(\overline{k} = k|Y : Y \to X\) is continuous and onto. So, \(Y\) is dense in \(EX\) and extremally disconnected. Let \(N\) be a nowhere dense subset of \(Y\). Suppose \(U = \text{int}_{EX}(\text{cl}_{EX}(N)) \neq \emptyset\). Then, as \(Y\) is dense, \(\emptyset \neq U \cap Y \subseteq \text{cl}_{EX}(N) \cap Y = \text{cl}_Y(N)\), contradicting that \(N\) is nowhere dense in \(Y\). Thus, \(N\) is nowhere dense in \(EX\). By 6.5d(2) in [PW], \(k[N] = \overline{k}[N]\) is nowhere dense in \(X\) and hence discrete in \(X\). Let \(p \in N\). There is an open set \(V\) in \(X\) such that \(V \cap \overline{k}[V] = \{k(p)\}\). So, \(N \cap \overline{k}[V] = \overline{k}^{-1}([p])\). But \(\overline{k}^{-1}([p])\) has only two points and there is an open set \(W\) in \(Y\) such that \(N \cap \overline{k}^{-1}[V] \cap W = \{p\}\). This shows that \(N\) is discrete in \(Y\) and \(Y\) is an \(\alpha\)-space. \(\square\)

We are ready to apply Proposition 29 to the regular \(\alpha\)-space \((\mathbb{R}, \delta(\mathbb{R}))\). First note that the covering function \(k : E\mathbb{R} \to (\mathbb{R}, \delta(\mathbb{R}))\) has the property that \(|k^{-1}(r)| \geq 2\) for each \(r \in \mathbb{R}\) since each open neighborhood filter is contained in two distinct \(\delta(\mathbb{R})\) open ultrafilters on \(\mathbb{R}\). Let \(Y \subseteq E\mathbb{R}\) have the property that \(|k^{-1}(r) \cap Y| = 2\) for each \(r \in \mathbb{R}\), and let \(Z \subset Y\) be such that \(|k^{-1}(r) \cap Z| = 1\) for each \(r \in \mathbb{R}\). Now, both \(Y\) and \(Z\) are dense in \(E\mathbb{R}\). By Proposition 29, \(Y\) is a Tychonoff \(\alpha\)-space. Let \(\sigma\) be the topology on \(Y\) generated by \(\tau(Y) \cup \{Z\}\). Clearly, \(\tau(Y) \subset \sigma\), and it is straightforward to show that \(\sigma_s = \tau(Y)\). The space \(Y\) is the desired space.

**5. A problem**

Since the property to be \(T_{2\frac{1}{2}}\)-closed does not pass to the regular open subspaces, we cannot use a result like Proposition 10 to prove a proposition similar to Proposition 23. So, we leave the following as an unsolved problem:
Conjecture. Let $X$ be an $\alpha T_{2\frac{1}{2}}$-space. Then $X$ is a minimal-$\alpha T_{2\frac{1}{2}}$-space if and only if $(X, \tau_s(X))$ is semiregular, $T_{2\frac{1}{2}}$-closed and $\tau(X) = \alpha \tau_s(X)$.

References