Some More Properties of $F_I$ and Regular $I$-Closed Sets in Ideal Topological Spaces

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Abstract. In 1964, Hayashi [8] defined and studied the notions of $\star$-dense in itself sets and $\star$-perfect subsets in ideal topological spaces. In 1999, Dontchev et al. [5] have studied the notion of Ideal resolvability through codense and completely codense ideal topological spaces. Recently, in the year 2004, Keskin, Noiri and Yüksel [12] have introduced and studied the concepts of $f_I$-sets and $f_I$-continuity. In this paper, we studied some more properties of $f_I$-sets and $f_I$-continuity with codense and completely codense ideals. We also continued the study of regular $I$-closed concepts.

2000 Mathematics Subject Classification: Primary 54A05, 54A10, Secondary 54C08

Key words and phrases: Codense and completely codense ideals, $\alpha$-set, semiopen set, preopen set, $I$-open set, $I$-locally closed set, $f_I$-set, regular $I$-closed set, semicontinuity, $f_I$-continuity.

1. Introduction

Ideals in topological spaces have been considered since 1930. This topic has won its importance by the paper of Vaidyanathaswamy [20]. An ideal $I$ on a topological space $(X, \tau)$ is a collection of subsets of $X$ which satisfies (i) $A \in I$ and $B \subset A$ implies $B \in I$ and (ii) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space $(X, \tau)$ with an ideal $I$ on $X$ and if $\wp(X)$ is the set of all subsets of $X$, a set operator $(\cdot)\ast: \wp(X) \rightarrow \wp(X)$, called a local function [13] of $A$ with respect to $\tau$ and $I$ is defined as follows: for $A \subset X$, $A^\ast(I, \tau) = \{ x \in X \mid U \cap A \notin I \text{ for every } U \in \tau(x) \}$ where $\tau(x) = \{ U \in \tau \mid x \in U \}$. We will make use of the basic facts about the local functions [9, Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^\ast(\cdot)$ for a topology $\tau^\ast(I, \tau)$, called the $\ast$-topology, finer than $\tau$.

Received: March 7, 2005; Revised: September 6, 2005.
is defined by $\text{cl}^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [20]. When there is no chance for confusion, we will simply write $A^*$ for $A^*(\mathcal{I}, \tau)$ and $\tau^*$ or $\tau^*(\mathcal{I})$ for $\tau^*(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on $X$, then $(X, \tau, \mathcal{I})$ is called an ideal space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in $(X, \tau)$. A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is $\tau^*$-closed [9] (resp. $\tau^*$-dense in itself [8], $\tau^*$-perfect [8]) if $A^* \subset A$ (resp. $A \subset A^*$, $A = A^*$). Clearly, $A$ is $\tau^*$-perfect if and only if $A$ is $\tau^*$-closed and $\tau^*$-dense in itself. In ideal topological spaces, $\mathcal{I}$-open sets [10], almost $\mathcal{I}$-open sets [1] (quasi $\mathcal{I}$-open sets [2]), $\mathcal{I}$-locally closed sets [4], $f_\mathcal{I}$-sets [12] and regular $\mathcal{I}$-closed sets [11] are some of the $\tau^*$-dense in itself sets. In this note, we discuss the properties of the $\tau^*$-dense in itself sets, namely $f_\mathcal{I}$-sets and regular $\mathcal{I}$-closed sets.

2. Preliminaries

By a space, we always mean a topological space $(X, \tau)$ with no separation properties assumed. If $A \subset X, \text{cl}(A)$ and $\text{int}(A)$ will, respectively, denote the closure and interior of $A$ in $(X, \tau)$ and $\text{cl}^*(A)$ and $\text{int}^*(A)$ will, respectively, denote the closure and interior of $A$ in $(X, \tau^*)$. An open subset $A$ of a space $(X, \tau)$ is said to be regular open if $A = \text{int} (\text{cl}(A))$. The complement of a regular open set is regular closed. The family of all regular open (resp. regular closed) set is denoted by $\text{RO}(X, \tau)$ (resp. $\text{RC}(X, \tau)$). A subset $A$ of a space $(X, \tau)$ is an $\alpha$-open [16] (resp. semiopen [14], preopen [15]) if $A \subset \text{int} (\text{cl}(\text{int}(A)))$ (resp. $A \subset \text{cl}(\text{int}(A))$, $A \subset \text{int}(\text{cl}(A)))$. The complement of a semiopen (resp. preopen) set is semiclosed (resp. preclosed). The family of all $\alpha$-open (resp. semiopen, preopen) sets in $(X, \tau)$ is denoted by $\tau^\alpha$ (resp. $\text{SO}(X, \tau)$, $\text{PO}(X, \tau)$). The smallest preclosed set containing $A$ is called the preclosure of $A$ and is denoted by $\text{pcl}(A)$. Also, $\text{pcl}(A) = A \cup \text{cl}(\text{int}(A))$ [3, Theorem 1.5(e)]. The largest preopen set contained in $A$ is called the preinterior of $A$ and is denoted by $\text{pint}(A)$. Also, $\text{pint}(A) = A \cap \text{int}(\text{cl}(A))$ [3, Theorem 1.5(f)]. $\tau^\alpha$ is a topology finer than $\tau$. The interior of $A$ in $(X, \tau^\alpha)$ is denoted by $\text{int}_\alpha(A)$ and $\text{int}_\alpha(A) = A \cap \text{int}(\text{cl}(\text{int}(A)))$ [3, Theorem 1.5(d)]. $\tau$ is said to be an $\alpha$-topology [16] if $\tau = \tau^\alpha$. Two topologies $\tau$ and $\sigma$ on $X$ is said to be $\alpha$-equivalent [16] if $\tau^\alpha = \sigma^\alpha$. Recall that, if two topologies on a set $X$ are $\alpha$-equivalent, then they have the same collection of regular open, semiopen, preopen, dense and nowhere dense sets. A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}$-open [10] if $A \subset \text{int}(A^*)$. The family of all $\mathcal{I}$-open sets is denoted by $\text{IO}(X, \tau)$. A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is $\mathcal{I}$-locally closed [4] if $A = G \cap V$, where $G$ is open and $V$ is $\tau^*$-perfect. A subset $A$ of $X$ is $\mathcal{I}$-locally closed if and only if $A = G \cap A^*$ for some open set $G$ [19, Theorem 2.2]. Clearly, every $\tau^*$-perfect set is $\mathcal{I}$-locally closed. Given an ideal space $(X, \tau, \mathcal{I})$, $\mathcal{I}$ is said to be compatible with respect to $\tau$ [9] (supercompact [20]), denoted by $\mathcal{I} \sim \tau$, if for every subset $A$ of $X$ and for each $x \in A$, there exists a neighborhood $U$ of $x$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. $\mathcal{I}$ is said to be codense [5] if $\tau \cap \mathcal{I} = \{\emptyset\}$. $\mathcal{I}$ is said to be completely codense [5] if $\text{PO}(X) \cap \mathcal{I} = \{\emptyset\}$. Every completely codense ideal is codense but not the converse [5]. The following lemmas will be useful in the sequel.

Lemma 2.1. [9, Theorem 6.1] Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.

(a) $\mathcal{I}$ is codense.
Lemma 2.2. [18, Theorem 5] Let \((X, \tau, \mathcal{I})\) be an ideal space. If \(A\) is \(*\)-dense in itself, then \(A^* = \text{cl}(A) = \text{cl}^*(A)\).

Lemma 2.3. [18, Corollary 4] If \(\mathcal{I}\) is a completely codense ideal of an ideal space \((X, \tau, \mathcal{I})\), then

(a) \(\tau \subset \tau^* \subset \tau^0\),
(b) \(SO(X, \tau) = SO(X, \tau^*) = SO(X, \tau^0)\),
(c) \((\tau^*)^0 = \tau^0\).

Lemma 2.4. [19, Theorem 2.1] Let \((X, \tau, \mathcal{I})\) be an ideal space and \(U\) and \(A\) be subsets of \(X\) such that \(A \subset U \subset A^*\). Then \(U\) is \(*\)-dense in itself, and \(U^*\) and \(A^*\) are \(*\)-perfect.

Lemma 2.5. [19, Theorem 2.15] If \((X, \tau, \mathcal{I})\) is an ideal space, then \(\mathcal{I}\) is completely codense if and only if \(PO(X, \tau) = IO(X, \tau)\).

3. More properties of codense ideals and completely codense ideals

The following Theorem 3.1 and its corollary give relationship between codense and completely codense ideals. Given a space \((X, \tau)\) and ideals \(\mathcal{I}\) and \(\mathcal{N}\) on \(X\), the extension of \(\mathcal{I}\) via \(\mathcal{N}\) [10], denoted by \(\mathcal{I} \star \mathcal{N}\), is the ideal given by \(\mathcal{I} \star \mathcal{N} = \{A \subset X \mid A^*(\mathcal{I}) \in \mathcal{N}\}\). In particular, \(\mathcal{I} \star \mathcal{N} = \{A \subset X \mid \text{int}(A^*(\mathcal{I})) = \emptyset\}\) is a compatible ideal containing both \(\mathcal{I}\) and \(\mathcal{N}\) and \(\mathcal{I} \star \mathcal{N}\) is usually denoted by \(\widehat{\mathcal{I}}\). Since \(\widehat{\mathcal{I}}\) is compatible, \((A^*(\widehat{\mathcal{I}}))^*(\mathcal{I}) = A^*(\mathcal{I})\) [9, Theorem 4.6(b)]. In Theorem 3.2 below, we discuss the relationship between the \(\alpha\)-sets of the topologies \(\tau\), \(\tau^*(\mathcal{I})\) and \(\tau^*(\widehat{\mathcal{I}})\).

Theorem 3.1. Let \((X, \tau, \mathcal{I})\) be an ideal space. Then \(\mathcal{I}\) is codense in \((X, \tau)\) if and only if \(\widehat{\mathcal{I}}\) is completely codense in \((X, \tau^*)\).

Proof. If \(\mathcal{I}\) is completely codense in \((X, \tau^*)\), then \(\mathcal{I}\) is codense in \((X, \tau^*)\) and so \(\mathcal{I}\) is codense in \((X, \tau)\). Conversely, suppose \(\mathcal{I}\) is codense in \((X, \tau)\). Let \(A \in PO(X, \tau^*) \cap \mathcal{I}\). A \(\subset PO(X, \tau^*) \cap \mathcal{I}\Rightarrow A \in PO(X, \tau^*)\) and \(A \in \mathcal{I}\). A \(\in PO(X, \tau^*) \Rightarrow A \subset\ \text{int}^*(\text{cl}^*(A))\). \(A \in \mathcal{I} \Rightarrow \text{int}^*(A) = \emptyset\), by Lemma 2.1(d) and \(A\) is \(\tau^*\)-closed, by [9, Lemma 2.7]. Therefore, \(\text{int}^*(\text{cl}^*(A)) = \text{int}^*(A) = \emptyset\) which implies that \(A = \emptyset\). Therefore, \(\mathcal{I}\) is completely codense in \((X, \tau^*)\). \(\square\)

Corollary 3.1. If \((X, \tau, \mathcal{I})\) is an ideal space, then the following are equivalent.

(a) \(\mathcal{I}\) is codense in \((X, \tau)\).
(b) \(\widehat{\mathcal{I}}\) is codense in \((X, \tau)\).
(c) \(\mathcal{I}\) is completely codense in \((X, \tau^*)\).
(d) \(\widehat{\mathcal{I}}\) is completely codense in \((X, \tau^*)\).

Proof. (a) and (b) are equivalent by [10, Theorem 3.5]. (b) and (c) are equivalent by Theorem 3.1. (a) and (d) are equivalent by Theorem 3.1. \(\square\)
Theorem 3.2. Let \((X, \tau, \mathcal{I})\) be an ideal space. Then

(a) \(\tau^*(\widetilde{\mathcal{I}}) = (\tau^*(\mathcal{I}))^\ast(\widetilde{\mathcal{I}})\).

(b) If \(\mathcal{I}\) is codense, then \((\tau^*(\mathcal{I}))^\ast = (\tau^*(\mathcal{I}))^\circ\).

(c) If \(\mathcal{I}\) is completely codense, then \((\tau^*(\mathcal{I}))^\circ = \tau^*(\mathcal{I})\).

Proof. (a) Since \(\mathcal{I} \subseteq \widetilde{\mathcal{I}}, \tau^*(\mathcal{I}) \subseteq \tau^*(\widetilde{\mathcal{I}})\) which implies that \((\tau^*(\mathcal{I}))^\ast(\widetilde{\mathcal{I}}) \subseteq (\tau^*(\widetilde{\mathcal{I}}))^\ast(\widetilde{\mathcal{I}})\) and so \((\tau^*(\mathcal{I}))^\ast(\widetilde{\mathcal{I}}) \subseteq (\tau^*(\mathcal{I}))^\circ(\widetilde{\mathcal{I}})\). Suppose \(A \in \tau^*(\widetilde{\mathcal{I}})\). For each \(x \in \mathcal{A}\), there exists \(U \in \tau\) and \(I \in \mathcal{I}\) such that \(x \in U - I \subseteq A\). Since \(U \in \tau^*(\mathcal{I})\), \(A \in (\tau^*(\mathcal{I}))^\ast(\widetilde{\mathcal{I}})\) and so \(\tau^*(\mathcal{I}) \subseteq (\tau^*(\mathcal{I}))^\ast(\widetilde{\mathcal{I}})\.

(b) If \(\mathcal{I}\) is codense, by Corollary 3.2, \(\mathcal{I}\) is completely codense in \((X, \tau^\ast)\) and so by [5, Theorem 4.13], \(\mathcal{I} \subseteq (\tau^*(\mathcal{I}))^\circ\). Therefore, \((\tau^*(\mathcal{I}))^\ast(\widetilde{\mathcal{I}}) \subseteq (\tau^*(\mathcal{I}))^\circ(\widetilde{\mathcal{I}})\).

(c) If \(\mathcal{I}\) is completely codense, by [18, Theorem 7(c)], \(\mathcal{I} = \mathcal{N}\) and so \(\tau^*(\mathcal{I}) \subseteq (\tau^*(\mathcal{I}))^\circ\). By Lemma 2.3(c), \((\tau^*(\mathcal{I}))^\circ = \tau^\circ\). Therefore, (c) follows. \(\square\)

We have the following.

Corollary 3.2. Let \((X, \tau, \mathcal{I})\) be an ideal space and \(\mathcal{I}\) be codense. Then \(\tau^*(\mathcal{I})\) and \(\tau^*(\mathcal{I})^\circ\) are \(\alpha\)-equivalent and so they have the same collection of regular open, semiopen, preopen, dense and nowhere dense sets.

Proof. By Theorem 3.2(b), \(\tau^*(\mathcal{I})\) and \(\tau^*(\mathcal{I})^\circ\) are \(\alpha\)-equivalent and so, they have the same collection of regular open, semiopen, preopen, dense and nowhere dense sets. \(\square\)

Corollary 3.3. Let \((X, \tau, \mathcal{I})\) be an ideal space and \(\mathcal{I}\) be completely codense. Then

(a) \(\tau^*(\mathcal{I})\) is an \(\alpha\)-topology.

(b) \(\tau^*(\mathcal{I}) = \tau^*(\mathcal{N})\), and

(c) \(\tau, \tau^*(\mathcal{I})\) and \(\tau^*(\mathcal{I})^\circ\) are \(\alpha\)-equivalent and so they have the same collection of regular open, semiopen, preopen, dense and nowhere dense sets.

Proof. (a) Since \((\tau^*(\mathcal{I}))^\circ = \tau^*(\mathcal{I})\), by Theorem 3.2(c), \(\tau^*(\mathcal{I})\) is an \(\alpha\)-topology.

(b) Since \(\tau^\circ = \tau^*(\mathcal{N})\) [9, Example 2.10], by Theorem 3.2(c), \(\tau^*(\mathcal{I}) = \tau^*(\mathcal{N})\).

(c) follows from Theorem 3.2(c). \(\square\)

4. Characterizations of \(f_{\mathcal{I}}\)-sets

A subset \(A\) of an ideal space \((X, \tau, \mathcal{I})\) is called an \(f_{\mathcal{I}}\)-set [12] if \(A \subseteq (\operatorname{int}(A))^\ast\). The family of all \(f_{\mathcal{I}}\)-sets in \((X, \tau, \mathcal{I})\) is denoted by \(\mathcal{F}(\tau, \mathcal{I})\). Clearly, if \(A\) is any non-empty \(f_{\mathcal{I}}\)-set, then \(\operatorname{int}(A) \neq \emptyset\) and if \(\mathcal{I}\) is not codense, then \(X\) is not an \(f_{\mathcal{I}}\)-set. In addition to this, \(F(\tau, \mathcal{I})\) has the following nice property.

Theorem 4.1. If \((X, \tau, \mathcal{I})\) is an ideal space, then \(\mathcal{F}(\tau, \mathcal{I}) \cap \mathcal{I} = \{\emptyset\}\).

Proof. If \(A \in \mathcal{F}(\tau, \mathcal{I}) \cap \mathcal{I}\), then \(A \subseteq \mathcal{F}(\tau, \mathcal{I})\) and \(A \in \mathcal{I}\). \(A \in \mathcal{I} \Rightarrow A^\ast = \emptyset\) and \(A \in \mathcal{F}(\tau, \mathcal{I}) \Rightarrow A \subseteq (\operatorname{int}(A))^\ast \subseteq A^\ast = \emptyset\). Therefore, \(A = \emptyset\) which completes the proof. \(\square\)
Every $f_2$-set is a semiopen set [12, Remark 2] but not the converse [12, Example 3]. The following Theorem 4.2 and its Corollary 4.1, characterizes codense ideals in terms of $f_2$-sets and Theorem 4.2 shows that for codense ideals, semiopen sets and $f_2$-sets coincide.

**Theorem 4.2.** Let $(X, \tau, \mathcal{I})$ be an ideal space, then the following are equivalent.

(a) $\mathcal{I}$ is codense.
(b) $SO(X, \tau) = \mathcal{F}(\tau, \mathcal{I})$.
(c) $\tau \subset \mathcal{F}(\tau, \mathcal{I})$.

Proof. (a) $\Rightarrow$ (b). If $A \in SO(X, \tau)$, then $A \subset \text{cl}(\text{int}(A)) = (\text{int}(A))^*$, by Lemma 2.1(c) and Lemma 2.2 and so $A \in \mathcal{F}(\tau, \mathcal{I})$. If $A \in \mathcal{F}(\tau, \mathcal{I})$, then $A \subset (\text{int}(A))^* = \text{cl}(\text{int}(A))$ and so $A \in SO(X, \tau)$.

(b) $\Rightarrow$ (c) is clear.

(c) $\Rightarrow$ (a). Follows from Theorem 4.1. □

**Corollary 4.1.** Let $(X, \tau, \mathcal{I})$ be an ideal space. Then the following are equivalent.

(a) $\mathcal{I}$ is codense.
(b) $SO(X, \tau^*) = \mathcal{F}(\tau^*, \mathcal{I})$.
(c) $\tau^* \subset \mathcal{F}(\tau^*, \mathcal{I})$.

Proof. Since $\mathcal{I}$ is codense in $(X, \tau)$ if and only if $\mathcal{I}$ is codense in $(X, \tau^*)$ by the remark below Theorem 6.1 of [9], the proof follows from Theorem 4.2. □

**Corollary 4.2.** [18, Theorem 1] Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is codense if and only if $SO(X, \tau) \cap \mathcal{I} = \{\emptyset\}$.

Since $\tau \subset \tau^*$, $\mathcal{F}(\tau, \mathcal{I}) \subset \mathcal{F}(\tau^*, \mathcal{I})$. The following Example 4.1, shows that the reverse direction is not true in general and Theorem 4.3 below shows that the two collection of sets are equal if the ideal $\mathcal{I}$ is completely codense.

**Example 4.1.** [12, Example 1] Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{e\}, \{a, d\}, \{a, c, d\}$, $X$ and $\mathcal{I} = \{\emptyset, \{b\}, \{d\}, \{b, d\}\}$. If $A = \{a\}$, then $\text{int}(A) = \emptyset$ and so $A \notin \mathcal{F}(\tau, \mathcal{I})$. Since $\text{int}^*(A) = A$ and $(\text{int}^*(A))^* = \{a, b, d\}$, $A \in \mathcal{F}(\tau^*, \mathcal{I})$.

**Theorem 4.3.** Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ be completely codense. Then $SO(X, \tau) = SO(X, \tau^*) = SO(X, \tau^\alpha) = \mathcal{F}(\tau, \mathcal{I}) = \mathcal{F}(\tau^*, \mathcal{I}) = \mathcal{F}(\tau^\alpha, \mathcal{I})$.

Proof. Since $\mathcal{I}$ is completely codense, by Lemma 2.3(b), $SO(X, \tau) = SO(X, \tau^*) = SO(X, \tau^\alpha)$. Since $\mathcal{N}$ is codense, $\mathcal{I} \subset \mathcal{N}$ and $\tau^*(\mathcal{N}) = \tau^\alpha$, by Theorem 4.2, $SO(X, \tau^\alpha) = \mathcal{F}(\tau^\alpha, \mathcal{I})$. Therefore, the proof follows from Theorem 4.2 and Corollary 4.1. □

**Corollary 4.3.** [12, Proposition 3(a)]. Let $(X, \tau, \mathcal{I})$ be an ideal space. If $\mathcal{I} = \{\emptyset\}$ or $\mathcal{N}$, then $SO(X, \tau) = \mathcal{F}(\tau, \mathcal{I})$.

**Corollary 4.4.** Let $(X, \tau, \mathcal{I})$ be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^\alpha = \mathcal{F}(X, \tau) \cap IO(X, \tau)$.

Proof. We know that $\tau^\alpha = SO(X, \tau) \cap PO(X, \tau)$ [17]. Since $\mathcal{I}$ is completely codense, by Lemma 2.5, $PO(X, \tau) = IO(X, \tau)$ and by Theorem 4.2, $SO(X, \tau) = \mathcal{F}(X, \tau)$. Therefore, the proof follows. □
Definition 4.1. A subset $A$ of an ideal space $(X, \tau, I)$ is called a regular $I$-closed set [11] if $A = (\text{int}(A))^\ast$. Every regular $I$-closed set is an $f_I$-set [12, Proposition 5].

The following Theorems 4.4 and 4.5, give some properties of $f_I$-sets.

**Theorem 4.4.** If $A$ is an $f_I$-set of an ideal space $(X, \tau, I)$, then

(a) $A$ and $\text{int}(A)$ are $\ast$-dense in itself.
(b) $(\text{int}(A))^\ast$ is $\ast$-dense in itself.
(c) $A^\ast = (\text{int}(A))^\ast = ((\text{int}(A))^\ast)^\ast$.
(d) $A^\ast$ is $\ast$-perfect and $I$-locally closed.
(e) $(\text{int}(A))^\ast$ is $\ast$-perfect and $I$-locally closed.
(f) $A^\ast(\overline{I})$ is $\ast$-dense in itself and $A^\ast = \text{cl}(\text{int}(A)) = \text{cl}(\text{int}(A^\ast)) = A^\ast(\overline{I})$.
(g) $A^\ast = A^\ast(\overline{I})$ is regular closed and $A \subset A^\ast(N)$.
(h) $A^\ast$ is regular $I$-closed and hence an $f_I$-set.

**Proof.** (a) $A$ is $\ast$-dense in itself by Corollary 1 of [12]. Since $\text{int}(A) \subset A \subset (\text{int}(A))^\ast$, $\text{int}(A)$ is $\ast$-dense in itself.

(b) Since $A \subset (\text{int}(A))^\ast \subset A^\ast$, by Lemma 2.4, $(\text{int}(A))^\ast$ is $\ast$-dense in itself.

(c) Since $A \subset (\text{int}(A))^\ast \subset A^\ast$, $(\text{int}(A))^\ast \subset (\text{int}(A))^\ast \subset A^\ast$ and so $A^\ast = ((\text{int}(A))^\ast)^\ast = (\text{int}(A))^\ast = (A^\ast)^\ast$.

(d) Since $(A^\ast)^\ast = A^\ast$, $A^\ast$ is $\ast$-perfect and so is $I$-locally closed.

(e) By (c), $(\text{int}(A))^\ast$ is $\ast$-perfect and so $I$-locally closed.

(f) Since $A^\ast = (\text{int}(A))^\ast \subset \text{cl}(\text{int}(A)) \subset \text{cl}(\text{int}(A^\ast)) = A^\ast(\overline{I})$ by [10, Theorem 3.2] and $A^\ast(\overline{I}) \subset A^\ast$, $A^\ast(\overline{I})$ is $\ast$-dense in itself and $A^\ast = \text{cl}(\text{int}(A)) = \text{cl}(\text{int}(A^\ast)) = A^\ast(\overline{I})$.

(g) Since $A^\ast(\overline{I})$ is regular closed by [18, Theorem 10(b)], by (f), $A^\ast$ is regular closed. Since $N \subset \overline{I}$, $A \subset A^\ast(N)$.

(h) Since $A \subset (\text{int}(A))^\ast \subset (\text{int}(A^\ast))^\ast \subset A^\ast$, we have $A^\ast = (\text{int}(A^\ast))^\ast$ and so $A^\ast$ is regular $I$-closed and so is an $f_I$-set. □

**Theorem 4.5.** Let $(X, \tau, I)$ be an ideal space and $A$ be an $f_I$-subset of $X$. Then

(a) $\text{pcl}(A) = \text{cl}(\text{int}(A)) = \text{cl}(A) = A^\ast$.
(b) $\text{pint}(A) = \text{int}_c(A)$.
(c) $\text{pint}(\text{pcl}(A)) = \text{int}(\text{pcl}(A)) = \text{int}(A^\ast)$.

**Proof.** (a) $A \in \mathcal{F}(\tau, I) \Rightarrow A \subset (\text{int}(A))^\ast \subset \text{cl}(\text{int}(A)) \Rightarrow A \cup \text{cl}(\text{int}(A)) = \text{cl}(\text{int}(A)) \Rightarrow \text{pcl}(A) = \text{cl}(\text{int}(A))$. By Theorem 4.4(f), $\text{pcl}(A) = A^\ast = \text{cl}(A)$.

(b) $\text{pint}(A) = A \cap \text{int}(\text{cl}(A)) = A \cap \text{int}(A^\ast)$, since $A$ is $\ast$-dense in itself. By (a),

$\text{pint}(A) = A \cap \text{int}(\text{cl}(A))$ and so $\text{pint}(A) = \text{int}(A)$.

(c) $\text{pint}(\text{pcl}(A)) = \text{pint}(\text{cl}(A))$, by (a) and so $\text{pint}(\text{pcl}(A)) = \text{cl}(A) \cap \text{int}(\text{cl}(A)) = \text{int}(\text{cl}(A)) = \text{int}(\text{pcl}(A)) = \text{int}(A^\ast)$. □

We recall the following.

**Definition 4.2.** A mapping $f : (X, \tau, I) \to (Y, \sigma)$ is said to be $f_I$-continuous [12] (resp. semicontinuous [14]) if for every $V \in \sigma$, $f^{-1}(V)$ is an $f_I$-set (resp. semipoint-open set).
Corollary 4.5. Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ be codense. Then $f: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $f_\mathcal{I}$-continuous if and only if $f$ is semicontinuous.

Proof. Follows from Theorem 4.2.

Corollary 4.6. Let $(X, \tau, \mathcal{I})$ be an ideal space where $\mathcal{I}$ be completely codense. Then the following are equivalent.

(a) $f: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $f_\mathcal{I}$-continuous.
(b) $f: (X, \tau^\star, \mathcal{I}) \to (Y, \sigma)$ is $f_\mathcal{I}$-continuous.
(c) $f: (X, \tau^\alpha, \mathcal{I}) \to (Y, \sigma)$ is $f_\mathcal{I}$-continuous.
(d) $f: (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is semicontinuous.
(e) $f: (X, \tau^\star, \mathcal{I}) \to (Y, \sigma)$ is semicontinuous.
(f) $f: (X, \tau^\alpha, \mathcal{I}) \to (Y, \sigma)$ is semicontinuous.

Proof. Proof follows from Theorem 4.3.

A subset $A$ of an ideal space $(X, \tau, \mathcal{I})$ is called an $\alpha - I$-open set [7] if $A \subset \text{int}(\text{cl}^\star(\text{int}(A)))$. $\alpha - I$-openness and $I$-openness are independent concepts [7, Remark 2.1]. In [18, Corollary 1(iii)], it was established that every $\star$-dense in itself, $\alpha - I$-open subset is $I$-open. The following Theorem 4.6, shows that the two kinds of sets are equivalent for the collection of $f_\mathcal{I}$-sets.

Theorem 4.6. Let $(X, \tau, \mathcal{I})$ be an ideal space and $A$ be an $f_\mathcal{I}$-subset of $X$. Then $A$ is $\alpha - I$-open if and only if $A$ is $I$-open.

Proof. Suppose $A$ is $\alpha - I$-open. Since $A$ is an $f_\mathcal{I}$-subset, $A$ is $\star$-dense in itself and so by [18, Corollary 1(iii)], $A$ is $I$-open. Conversely, suppose $A$ is $I$-open. Then $A \subset \text{int}(A^\star) = \text{int}(\text{int}(A))$ by Theorem 4.4(c), and so $A \subset \text{int}(\text{cl}^\star(\text{int}(A)))$. Therefore, $A$ is $\alpha - I$-open.

We end this section with the following characterization of $f_\mathcal{I}$-sets in terms of open sets.

Theorem 4.7. Let $(X, \tau, \mathcal{I})$ be an ideal space. Then $A$ is an $f_\mathcal{I}$-subset of $X$ if and only if there exists an open set $G$ such that $G \subset A \subset G^\star$.

Proof. Suppose $A$ is an $f_\mathcal{I}$-subset of $X$. Let $G = \text{int}(A)$. Then $G$ is the required open set such that $G \subset A \subset G^\star$. Conversely, suppose there is an open set $G$ such that $G \subset A \subset G^\star$. Now $G \subset A \Rightarrow G \subset \text{int}(A) \Rightarrow G^\star \subset (\text{int}(A))^\star \Rightarrow A \subset (\text{int}(A))^\star$ and so $A$ is an $f_\mathcal{I}$-subset.

Corollary 4.7. If $A$ is an $f_\mathcal{I}$-subset of an ideal space $(X, \tau, \mathcal{I})$, then there exists an open set $G \subset A$ such that $A^\star = G^\star$.

5. Properties of regular $\mathcal{I}$-closed sets

We will denote the family of all regular $\mathcal{I}$-closed sets in $(X, \tau, \mathcal{I})$ by $\mathcal{R}(\tau, \mathcal{I})$. If the ideal $\mathcal{I}$ is not codense, then $X$ is regular closed in $(X, \tau, \mathcal{I})$ but not regular $\mathcal{I}$-closed and so regular closed sets need not be regular $\mathcal{I}$-closed. But every regular $\mathcal{I}$-closed set is a regular closed set by Theorem 5.4(e) below. The easy proof of the following
Theorems 5.1 and 5.2 are omitted. Theorem 5.2 below gives a characterization of codense ideals.

**Theorem 5.1.** If $(X, \tau, I)$ is an ideal space, then $R(\tau, I) \cap I = \{\emptyset\}$.

**Theorem 5.2.** Let $(X, \tau, I)$ be an ideal space. Then $I$ is codense if and only if $X$ is regular $I$-closed.

**Theorem 5.3.** If $(X, \tau, I)$ is an ideal space where $I$ is codense, then $R(\tau, I) = RC(X, \tau)$.

**Proof.** $A \in R(\tau, I) \iff A = (\text{int}(A))^* \iff A = \text{cl}(\text{int}(A))$, since $I$ is codense $\iff A \in RC(X, \tau)$. \hfill $\Box$

**Corollary 5.1.** If $(X, \tau, I)$ is an ideal space where $I$ is codense, then the following are equivalent.

(a) $A \in RC(X, \tau)$.

(b) $A \in R(\tau, I)$.

(c) $A \in F(\tau, I)$ and $A$ is $\tau^*$-closed.

(d) $A \in SO(\tau, \tau)$ and $A$ is $\tau^*$-closed.

**Proof.** Proof follows from Theorem 5.3, [12, Proposition 5] and Theorem 4.2. \hfill $\Box$

The following Theorem 5.4 gives some properties of regular $I$-closed sets. Also, it is established that every regular $I$-closed set is $I$-locally closed.

**Theorem 5.4.** If $A$ is a regular $I$-closed set of an ideal space $(X, \tau, I)$, then

(a) $A$ and $\text{int}(A)$ are $*$-dense in itself.

(b) $A^* = (\text{int}(A))^* = (\text{int}(A))^* = A$.

(c) $A$ is $*$-perfect and $I$-locally closed.

(d) $(\text{int}(A))^*$ is $*$-perfect and $I$-locally closed.

(e) $A = \text{cl}(\text{int}(A)) = A^*(I)$ and so $A$ is regular closed.

**Proof.** (a) Since $\text{int}(A) \subset A = (\text{int}(A))^* \subset A^*$, $\text{int}(A)$ and $A$ are $*$-dense in itself.

(b) Since $A = (\text{int}(A))^* \subset A^*$, $A^* = ((\text{int}(A))^*)^* \subset (\text{int}(A))^* = A \subset A^*$ and so $A^* = ((\text{int}(A))^*)^* = (\text{int}(A))^* = A$.

(c) Since $A = A^*$, $A$ is $*$-perfect and so is $I$-locally closed.

(d) By (b), $(\text{int}(A))^*$ is $*$-perfect and so $I$-locally closed.

(e) Since $A = (\text{int}(A))^* \subset \text{cl}(\text{int}(A)) = \text{cl}(\text{int}(A^*)) = A^*(I)$ by Theorem 3.2 of [10] and $A^*(I) \subset A^* = A$, $A = \text{cl}(\text{int}(A)) = A^*(I)$ and so $A$ is regular closed. \hfill $\Box$

We end this section with the following characterization of regular $I$-closed sets in terms of open sets.

**Theorem 5.5.** Let $(X, \tau, I)$ be an ideal space. Then $A$ is a regular $I$-closed subset of $X$ if and only if there exists an open set $G$ such that $G \cap A = G^*$.

**Proof.** Suppose $A$ is a regular $I$-closed subset of $X$. Let $G = \text{int}(A)$. Then $G$ is the required open set such that $G \subset A = G^*$. Conversely, suppose that there is an open set $G$ such that $G \subset A = G^*$. Now $G \subset A \Rightarrow G \subset \text{int}(A) \Rightarrow G^* \subset (\text{int}(A))^* \Rightarrow A \subset (\text{int}(A))^* \subset A^* = G^* = A$. Therefore, $A$ is regular $I$-closed. \hfill $\Box$

**Acknowledgement.** The authors sincerely thank the referees for their valuable suggestions.
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