THE STRUCTURE OF SOME CLASSES OF 3-DIMENSIONAL NORMAL ALMOST CONTACT METRIC MANIFOLDS

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Abstract. The object of the present paper is to study $\xi$–projectively flat and $\phi$–projectively flat 3-dimensional normal almost contact metric manifolds. An illustrative example is given.

1. Introduction

Let $M$ be an almost contact manifold and $(\phi, \xi, \eta)$ its almost contact structure. This means, $M$ is an odd-dimensional differentiable manifold and $\phi, \xi, \eta$ are tensor fields on $M$ of types (1, 1), (1, 0), (0, 1) respectively, such that

\[ \phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0. \] (1)

Let $R$ be the real line and $t$ a coordinate on $R$. Define an almost complex structure $J$ on $M \times R$ by

\[ J(X, \frac{df}{dt}) = (\phi X - \lambda \xi, \eta(X) \frac{df}{dt}), \]

where the pair $(X, \frac{df}{dt})$ denotes a tangent vector to $M \times R$, $f$ is a smooth function on $M \times R$, $X$ and $\frac{df}{dt}$ being tangent to $M$ and $R$ respectively.

$M$ with the structure $(\phi, \xi, \eta)$ is said to be normal if the structure $J$ is integrable [2], [3]. The necessary and sufficient condition for $(\phi, \xi, \eta)$ to be normal is

\[ [\phi, \phi] + 2d \eta \otimes \xi = 0, \]

where the pair $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$ defined by

\[ [\phi, \phi](X, Y) = [\phi X, \phi Y] + \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y], \]

for any $X, Y \in \mathcal{T}(M)$. We say that the form $\eta$ has rank $r = 2s$ if $(d\eta)^s \neq 0$, and $\eta \wedge (d\eta)^s = 0$, and has rank $r = 2s + 1$ if $\eta \wedge (d\eta)^s = 0$ and $(d\eta)^{s+1} = 0$. We also say that $r$ is the rank of the structure $(\phi, \xi, \eta)$.

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A Riemannian metric $g$ on $M$ satisfying the condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

(1.2)

for any $X, Y \in T(M)$ is said to be compatible with the structure $(\phi, \xi, \eta)$. If $g$ is such a metric, then the quadruple $(\phi, \xi, \eta, g)$ is called an almost contact metric (shortly a.c.m.) structure on $M$ and $M$ is an (a.c.m.) manifold. On such a manifold we also have $\eta(X) = g(X, \xi)$, for any $X \in T(M)$ and we can always define the 2-form $\Phi$ by $\Phi(X, Y) = g(X, \phi Y)$, where $X, Y \in T(M)$.

It is no hard to see that if $\dim M = 3$, then two Riemannian metrics $g$ and $g'$ are compatible with the same almost contact structure $(\phi, \xi, \eta)$ on $M$ if and only if $g' = g + (1 - \sigma)\eta \otimes \eta$, for a certain positive function $\sigma$ on $M$.

A normal (a.c.m.) structure $(\phi, \xi, \eta, g)$ satisfying additionally the condition $d\eta = \Phi$ is called Sasakian. Of course, any such structure on $M$ has rank 3. Also a normal almost contact metric structure satisfying the condition $d\Phi = 0$ is said to be quasi-Sasakian [4]. Contact metric manifolds have been studied by several authors ([1], [7], [12]).

Also if we consider $\tilde{M}$ be a complex $n$-dimensional Kaehler manifold and $M$ a real hypersurface of $\tilde{M}$. We denote by $\tilde{g}$ and $\tilde{J}$ a Kaehler metric tensor and its Hermitian Structure tensor, respectively. For any vector field $X$ tangent to $M$, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where $\phi$ is a $(1,1)$-type tensor field, $\eta$ is a 1-form and $\xi$ is a unit vector field on $M$. The induced Riemannian metric on $M$ is denoted by $g$. Then by the properties of $(\tilde{g}, \tilde{J})$, we see that the structure $(\phi, \xi, \eta, g)$ is an almost contact metric structure on $M$. Real hypersurfaces of a complex manifold have been studied by ([15], [18]) and many others.

In a recent paper [13], Olszak studied the curvature properties of normal almost contact manifold of dimension 3 with several examples. De, Yildiz and Funda [9] studied locally $\phi$-symmetric normal (a.c.m.) manifolds of dimension 3. Also De and Kalam [8] recently characterized certain curvature conditions on 3-dimensional normal almost contact manifolds. Since at each point $P \in M$, the tangent space $T_P(M)$ can be decomposed into the direct sum $T_P(M) = \phi(T_P(M)) \oplus \{\xi\}$, where $\{\xi_P\}$ is the 1-dimensional linear subspace of $T_P(M)$ generated by $\xi_P$, the conformal curvature tensor $C$ is a map $C : T_P(M) \times T_P(M) \times T_P(M) \to \phi(T_P(M)) \oplus \{\xi\}$, $P \in M$. One has the following well known particular cases: (1) the projection of the image of $C$ in $\phi(T_P(M))$ is zero; (2) the
projection of the image of \( C \) in \( \{ \xi, \phi \} \) is zero; and (3) the projection of the image of \( C \left| \phi(T_p(M)) \right| \) in

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\( \phi(T_p(M)) \) is zero. An (a.c.m.) manifold satisfying the cases (1), (2) and (3) is said to be conformally symmetric [19], \( \xi \) – conformally flat [20] and \( \phi \) – conformally flat [6] respectively.

Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view. Let \( M \) be a \( n \)–dimensional Riemannian manifold. If there exist one-to-one correspondence between each coordinate neighborhood of \( M \) and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then \( M \) is said to be locally projectively flat. For \( n \geq 3 \), \( M \) is locally projectively flat if and only if the well known projective curvature tensor \( P \) vanishes. Here \( P \) is defined by [11]

\[
P(X, Y)Z = R(X, Y)Z - \frac{1}{n - 1} \{ S(Y, Z)X - S(X, Z)Y \},
\]

for \( X, Y, Z \in T(M) \), where \( R \) is the curvature tensor and \( S \) is the Ricci tensor. In fact, \( M \) is projectively flat (that is \( P = 0 \)) if and only if the manifold is of constant curvature (pp. 84-85 of [16]). Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

The present paper is devoted to study \( \xi \) – projectively flat and \( \phi \) – projectively flat normal(a.c.m.) metric manifold of dimension 3. After preliminaries in section 3, we prove that a compact 3-dimensional normal (a.c.m.) manifold is \( \xi \) – projectively flat if and only if the manifold is \( \beta \) – Sasakian. In the next section, it is proved that a 3-dimensional normal (a.c.m.) manifold is \( \phi \) – projectively flat if and only if it is an Einstein manifold provided \( \alpha, \beta \) = constant. Finally we cited an example of a \( \phi \) – projectively flat normal almost contact metric manifold.

2. Preliminaries

For a normal (a.c.m.) structure \( (\phi, \xi, \eta, g) \) on \( M \), we have [13]

\[
\nabla_X \xi = \alpha \{ X - \eta(X) \xi \} - \beta \phi X,
\]

where \( 2 \alpha = \text{div} \xi \) and \( 2 \beta = \text{tr}(\phi \nabla \xi) \), \( \text{div} \xi \) is the divergence of \( \xi \) defined by

\[
\text{div} \xi = \text{trace}(X \rightarrow \nabla_X \xi) \quad \text{and} \quad \text{tr}(\phi \nabla \xi) = \text{trace}(X \rightarrow \phi \nabla_X \xi).
\]

As a consequence of (2.1) we have

\[
(\nabla_X \phi)(Y) = g(\phi \nabla_X \xi, Y) \xi - \eta(Y) \phi \nabla_X \xi
= \alpha \{ g(\phi X, Y) \xi \} - \eta(Y) \phi X \} + \beta \{ g(X, Y) \xi \} - \eta(Y)X \},
\]

(2.2)
\[ R(X, Y)\xi = \{Y\alpha + (\alpha^2 - \beta^2)\eta(Y)\}\phi^2 X - \{X\alpha + (\alpha^2 - \beta^2)\eta(X)\}\phi^2 Y \]
\[ + \{Y\beta + 2\alpha\beta\eta(Y)\}\phi X - \{X\beta + 2\alpha\beta\eta(X)\}\phi Y, \] (2.3)

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\[ S(X, Y) = \left(\frac{r}{2} + \xi\alpha + \alpha^2 + \beta^2\right)g(X, Y) - \left(\frac{r}{2} + \xi\alpha + 3(\alpha^2 - \beta^2)\right)\eta(X)\eta(Y) \]
\[ - (\eta(Y)X\alpha + \eta(X)Y\alpha) - \{\eta(Y)(\phi X)\beta + \eta(X)(\phi Y)\beta\}, \] (2.4)
\[ S(X, \xi) = -Y\alpha - (\phi(Y)\beta - \{\xi\alpha + 2(\alpha^2 - \beta^2)\}\eta(Y), \] (2.5)
\[ \xi\beta + 2\alpha\beta = 0, \] (2.6)

where \( R \) denotes the curvature tensor and \( S \) is the Ricci tensor.

On the other hand, the curvature tensor in a 3-dimensional Riemannian manifold always satisfies
\[ \tilde{R}(X, Y, Z, W) = g(X, W)S(Y, Z) - g(X, Z)S(Y, W) + g(Y, Z)S(X, W) \]
\[ - g(Y, W)S(X, Z) - \frac{r}{2}[g(X, W)g(Y, Z) - g(X, Z)g(Y, W)], \] (2.7)

where \( \tilde{R}(X, Y, Z, W) = g(R(X, Y)Z, W) \) and \( r \) is the scalar curvature. From (2.3) we can derive that
\[ \tilde{R}(\xi, Y, Z, \xi) = -(\xi\alpha + \alpha^2 + \beta^2)g(\phi Y, \phi Z) - (\xi\beta + 2\alpha\beta)g(\phi Y, \phi Z). \] (2.8)

By (2.5), (2.7) and (2.8) we obtain for \( \alpha, \beta = \text{constant}, \)
\[ S(Y, Z) = \left(\frac{r}{2} + \alpha^2 - \beta^2\right)g(\phi Y, \phi Z) - 2(\alpha^2 - \beta^2)\eta(Y)\eta(Z). \] (2.9)

From (2.6) it follows that if \( \alpha, \beta = \text{constant}, \) then the manifold is either \( \beta - \text{Sasakian}, \) or \( \alpha - \text{Kenmotsu [10]} \) or cosymplectic [2].

**Proposition 2.1.** A 3-dimensional normal almost contact metric manifold with \( \alpha, \beta = \text{constant} \) is either \( \beta - \text{Sasakian}, \) or \( \alpha - \text{Kenmotsu or cosymplectic}. \)

**Definition 1.** An almost \( C(\lambda) - \text{manifold} \) \( M \) is an almost co-Hermitian manifold such that the Riemannian curvature tensor satisfies the following property: there exist \( \lambda \in R \) such that for all \( \xi, Y, Z, W \in T(M) \)
\[ \tilde{R}(X, Y, Z, W) = \tilde{R}(X, Y, \phi Z, \phi W) + \{-g(X, Z)g(Y, W) + g(X, W)g(Y, Z) \]
\[ + g(X, \phi Z)g(Y, \phi W) - g(X, \phi W)g(Y, \phi Z)\}. \]
A normal almost $C(\lambda)$–manifold is a $C(\lambda)$–manifold. If we take $\lambda = -\alpha^2$ for $\alpha > 0$, then we get $C(\alpha^2)$–manifold. We note that $\beta$–Sasakian manifold are quasi-Sasakian [4]. They provide examples of $C(\lambda)$–manifolds with $\lambda \geq 0$. An $\alpha$–Kenmotsu manifold is a $C(\alpha^2)$–manifold [10]. Cosymplectic manifolds provide a natural setting for time dependent mechanical systems as they are locally product of a Kaehler manifold and a real line or a circle [5].

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3. 3-dimensional $\xi$–projectively flat normal almost contact metric Manifolds

$\xi$–conformally flat K–contact manifolds have been studied by Zhen, Cabrerizo and Fernandez [20]. In this section we study $\xi$–projectively flat normal (a.c.m.) manifold. Analogous to the definition of $\xi$–conformally flat (a.c.m.) manifold we define $\xi$–projectively flat (a.c.m.) manifold.

**Definition 3.1** A normal almost contact metric manifold $M$ is called $\xi$–projectively flat if the condition $P(X, Y)\xi = 0$ holds on $M$, where projective curvature tensor $P$ is defined by (1.3).

Putting $Z = \xi$ in (1.3) and using (2.3) and (2.5), we get

$$P(X, Y)\xi = -\frac{1}{2}\{(Y\alpha)X - (X\alpha)Y\} + \{(Y\alpha)\eta(X) - (X\alpha)\eta(Y)\} + (Y\beta)\phi X - (X\beta)\phi Y$$

$$+ 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) + \frac{1}{2}[(\eta(Y)\beta X - (\phi X)\beta Y + (\xi\alpha)\{\eta(Y)X - \eta(X)Y\}].$$

(3.1)

Now assume that $M$ is a compact 3-dimensional $\xi$–projectively flat normal (a.c.m.) manifold. Then from (3.1) we can write

$$-\frac{1}{2}\{(Y\alpha)X - (X\alpha)Y\} + \{(Y\alpha)\eta(X) - (X\alpha)\eta(Y)\} + (Y\beta)\phi X - (X\beta)\phi Y$$

$$+ 2\alpha\beta(\eta(Y)\phi X - \eta(X)\phi Y) + \frac{1}{2}[(\eta(Y)\beta X - (\phi X)\beta Y + (\xi\alpha)\{\eta(Y)X - \eta(X)Y\}] = 0.$$

(3.2)

Putting $Y = \xi$ in (3.2) and using (2.6), we obtain

$$(X\alpha)\xi + (\phi X)\beta\xi - (\xi\alpha)\eta(X)\xi = 0$$

which implies

$$(X\alpha) + (\phi X)\beta - (\xi\alpha)\eta(X) = 0.$$  (3.3)

Now (3.3) can be written as

$$(X\alpha) + g(grad\beta, \phi X) - (\xi\alpha)\eta(X) = 0.$$  (3.4)
Differentiating (3.4) covariantly along $Y$, we get
\[
\nabla_Y(X\alpha) + g(\nabla_Y \text{grad} \beta, \phi X) + g(\text{grad} \beta, (\nabla_Y \phi)X) - Y(\xi\alpha)\eta(X) - (\xi\alpha)(\nabla_Y \eta)(X) = 0. \tag{3.5}
\]

Hence, by antisymmetrization with respect to $X$ and $Y$, we have from (3.5)

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\[
g(\nabla_Y \text{grad} \beta, \phi X) - g(\nabla_X \text{grad} \beta, \phi Y) + g(\text{grad} \beta, (\nabla_Y \phi)X) - g(\text{grad} \beta, (\nabla_X \phi)Y) - Y(\xi\alpha)\eta(X) + X(\xi\alpha)\eta(Y) - (\xi\alpha)\{(\nabla_Y \eta)(X) - (\nabla_X \eta)(Y)\} = 0.
\]

This implies
\[
g(\nabla_Y \text{grad} \beta, \phi X) - g(\nabla_X \text{grad} \beta, \phi Y) + \{(\nabla_Y \phi)X\beta - (\nabla_X \phi)Y\beta\} - Y(\xi\alpha)\eta(X) + X(\xi\alpha)\eta(Y) + 2(\xi\alpha)d\eta(X,Y) = 0. \tag{3.6}
\]

Using (2.2) and $d\eta = \beta\phi$ [13], (3.6) yields
\[
g(\nabla_Y \text{grad} \beta, \phi X) - g(\nabla_X \text{grad} \beta, \phi Y) + \{2\alpha g(\phi Y, X)\xi - \alpha(\eta(X)\phi Y - \eta(Y)\phi X) - \beta(\eta(X)Y - \eta(Y)X)\beta - Y(\xi\alpha)\eta(X) + X(\xi\alpha)\eta(Y) + 2(\xi\alpha)\Phi(X,Y) = 0. \tag{3.7}
\]

Let $\{e_1, e_2, \xi\}$ be an orthonormal $\phi$–basis where $\phi e_1 = -e_2$ and $\phi e_2 = e_1$. Taking $Y = e_1$ and $X = e_2$ in (3.7), we find that
\[
g(\nabla_{e_1} \text{grad} \beta, e_1) + g(\nabla_{e_2} \text{grad} \beta, e_2) = 2\alpha(\xi\beta) + 2\beta(\xi\alpha). \tag{3.8}
\]

On the other hand (2.6) yields $g(\text{grad} \beta, \xi) = -2\alpha\beta$, whence by covariant differentiation we get, on account of (2.1)
\[
g(\nabla_\xi \text{grad} \beta, \xi) = -2\alpha(\xi\beta) - 2\beta(\xi\alpha). \tag{3.9}
\]

Denoting by $\Delta$ the Laplacian defined by $\Delta = \text{divgrad}$, in view of (3.8) and (3.9) we have $\Delta\beta = 0$. Since $M$ is compact, $\beta$ is a constant.

Now if $\beta \neq 0$, (2.6) implies $\alpha = 0$. This implies $M$ is a $\beta$–Sasakian manifold. Conversely, if $M$ is a $\beta$–Sasakian manifold, then from (3.1) it is easy to see that $P(X,Y)\xi = 0$. Hence we can state the following:

**Theorem 3.1.** A compact 3-dimensional normal almost contact metric manifold is $\xi$–projectively flat if and only if it is a $\beta$–Sasakian manifold.
4. 3-dimensional $\phi$–projectively flat normal almost contact metric manifolds

Analogous to the definition of $\phi$–conformally flat contact metric manifold [6], we define $\phi$–projectively flat normal almost contact metric manifold. In this connection we can mention the work of Ozgur [14] who has studied $\phi$–projectively flat Lorentzian Para-Sasakian manifolds.

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Definition 4.1 A 3-dimensional normal almost contact metric manifold satisfying the condition

$$\phi^2 P(\phi X, \phi Y)\phi Z = 0$$

is called $\phi$–Projectively flat.

Let us assume that $M$ is a 3-dimensional $\phi$–projectively flat normal (a.c.m.) manifold. It can be easily seen that $\phi^2 P(\phi X, \phi Y)\phi Z = 0$ holds if and only if $g(P(\phi X, \phi Y)\phi Z, \phi W) = 0$, for $X, Y, Z, W \in T(M)$.

Using (1.3) and (1.1), $\phi$–projectively flat means

$$g(R(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2} \{S(\phi Y, \phi Z)g(\phi X, \phi W) - S(\phi X, \phi Z)g(\phi Y, \phi W)\}. \quad (4.1)$$

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of the vector fields in $M$ and using the fact that $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis. Putting $X = W = e_i$ in (4.1) and summing up with respect to $i$, then we have

$$\sum_{i=1}^{2} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{1}{2} \sum_{i=1}^{2} \{S(\phi Y, \phi Z)g(\phi X, \phi W) - S(\phi X, \phi Z)g(\phi Y, \phi W)\}. \quad (4.2)$$

It can be easily verified that

$$\sum_{i=1}^{2} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + (\xi \alpha + \alpha^2 - \beta^2)g(\phi Y, \phi Z),$$

$$\sum_{i=1}^{2} g(\phi e_i, \phi e_i) = 2,$$

$$\sum_{i=1}^{2} S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = S(\phi Y, \phi Z).$$

So using (1.2) and (2.4), the equation (4.2) becomes
which gives \( r = -6(\xi \alpha + \alpha^2 - \beta^2) \). So we state the following:

**Proposition 4.1.** The scalar curvature \( r \) of a 3-dimensional \( \phi \)-projectively flat normal almost contact metric manifold is \(-6(\xi \alpha + \alpha^2 - \beta^2)\).

Also if \( r = -6(\xi \alpha + \alpha^2 - \beta^2) \), it follows from (2.4) that the manifold is an Einstein manifold provided \( \alpha, \beta = \text{constant} \). Hence we can state the following:

**Proposition 4.2.** A 3-dimensional \( \phi \)-projectively flat normal almost contact metric manifold is an Einstein manifold, provided \( \alpha, \beta = \text{constant} \).

It is known [17] that a 3-dimensional Einstein manifold is a manifold of constant curvature. Also \( M \) is projectively flat if and only if it is of constant curvature [16]. Now trivially, projectively flatness implies \( \phi \)-projectively flat. Hence using Proposition 4.2 we can state the following:

**Theorem 4.1.** A 3-dimensional normal almost contact metric manifold is \( \phi \)-projectively flat if and only if it is an Einstein manifold, provided \( \alpha, \beta = \text{constant} \).

### 5. Example of a 3-dimensional normal almost contact metric Manifold

We consider the 3-dimensional manifold \( M = \{(x, y, z) \in R^3, z \neq 0\} \), where \( (x, y, z) \) are standard coordinate of \( R^3 \).

The vector fields
\[
e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = z \frac{\partial}{\partial z}
\]
are linearly independent at each point of \( M \).

Let \( g \) be the Riemannian metric defined by
that is, the form of the metric becomes

\[ g = \frac{dx^2 + dy^2 + dz^2}{z^2}. \]

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Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z,e_3) \) for any \( Z \in T(M) \).

Let \( \phi \) be the (1, 1) tensor field defined by

\[ \phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0. \]

Then using the linearity of \( \phi \) and \( g \), we have

\[ \eta(e_3) = 1, \]

\[ \phi^2 Z = -Z + \eta(Z)e_3, \]

\[ g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W), \]

for any \( Z, W \in T(M) \).

Then for \( e_3 = \xi \) the structure \((\phi, \xi, \eta, g)\) defines an almost contact metric structure on \( M \).

Let \( \nabla \) be the Levi-Civita connection with respect to the metric \( g \). Then we have

\[ [e_1, e_3] = e_1 e_3 - e_3 e_1 = z \frac{\partial}{\partial x} (z \frac{\partial}{\partial z}) - z \frac{\partial}{\partial z} (z \frac{\partial}{\partial x}) = z^2 \frac{\partial^2}{\partial x \partial z} - z \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} = -e_1. \]

Similarly, \([e_1, e_2] = 0\) and \([e_2, e_3] = -e_2\).

The Riemannian connection \( \nabla \) of the metric \( g \) is given by

\[ 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]), \quad (5.1) \]
which is known as Koszul’s formula.

Using (5.1) we have

\[ 2g(\nabla_{e_1} e_3, e_1) = -2g(e_1, e_1) = 2g(-e_1, e_1). \]  

(5.2)

Again by (5.1)

\[ 2g(\nabla_{e_1} e_2, e_2) = 0 = 2g(-e_1, e_2). \]  

(5.3)

and

\[ 2g(\nabla_{e_1} e_3, e_3) = 0 = 2g(-e_1, e_3). \]  

(5.4)

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From (5.2), (5.3) and (5.4) we obtain

\[ 2g(\nabla_{e_1} e_3, X) = 2g(-e_1, X). \]  

for all \( X \in T(M) \).

Thus \( \nabla_{e_1} e_3 = -e_1 \).

Therefore, (5.1) further yields

\[
\begin{align*}
\nabla_{e_1} e_3 &= -e_1, \\
\nabla_{e_1} e_2 &= 0, \\
\nabla_{e_1} e_1 &= e_3,
\end{align*}
\]

(5.5)

(5.5) tells us that the manifold satisfies (2.1) for \( \alpha = -1 \) and \( \beta = 0 \) and \( \xi = e_3 \). Hence the manifold is a normal almost contact metric manifold with \( \alpha, \beta = \) constants.

It is known that

\[ R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \]  

(5.6)

With the help of the above results and using (5.6) it can be easily verified that

\[
\begin{align*}
R(e_1, e_2)e_3 &= 0, \\
R(e_2, e_3)e_3 &= -e_2, \\
R(e_1, e_3)e_3 &= -e_1, \\
R(e_1, e_2)e_3 &= -e_1, \\
R(e_2, e_3)e_2 &= e_3, \\
R(e_1, e_3)e_2 &= 0, \\
R(e_1, e_2)e_1 &= e_2, \\
R(e_2, e_3)e_1 &= 0, \\
R(e_1, e_3)e_1 &= e_3.
\end{align*}
\]

From the above expressions of the curvature tensor we obtain

\[ S(e_1, e_1) = g(R(e_1, e_2)e_2, e_1) + g(R(e_1, e_3)e_3, e_1) = -2. \]

Similarly, we have

\[ S(e_2, e_2) = 0. \]

Therefore

\[ r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = -6. \]
We note that $\alpha, \beta$ and $r$ are all constants. It is sufficient to check

$$S(e_i, e_j) = -2 = -2(\alpha^2 - \beta^2)g(e_i, e_j),$$

for all $i = 1, 2, 3$ and $\alpha = -1$ and $\beta = 0$. Hence $M$ is an Einstein manifold. Therefore $M$ is $\phi-$projectively flat. Thus Theorem 4.1 is verified.

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