A Family of Integral Operators Preserving Subordination and Superordination

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Abstract

The main purpose of the present paper is to investigate some subordination-preserving and superordination-preserving properties of a certain family of integral operators. Several sandwich-type results associated with this family of integral operators are also derived.

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1. Introduction

Let \( \mathcal{A} \) denote the class of functions of the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]

which are analytic in the open unit disk

\[
\mathbb{U} := \{ z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1 \}.
\]

Let \( \mathcal{H}(\mathbb{U}) \) be the linear space of all analytic functions in \( \mathbb{U} \). For a positive integer number \( n \) and \( a \in \mathbb{C} \), we let

\[
\mathcal{H}[a,n] := \{ f \in \mathcal{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \}.
\]

Let \( f, g \in \mathcal{A} \), where \( f \) is given by (1.1) and \( g \) is defined by

\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k.
\]

Then the Hadamard product (or convolution) \( f \ast g \) of the functions \( f \) and \( g \) is defined by

\[
(f \ast g)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k =: (g \ast f)(z).
\]
For two functions \( f \) and \( g \), analytic in \( U \), we say that the function \( f \) is subordinate to \( g \) in \( U \), and write 
\[ f(z) \prec g(z), \]
if there exists a Schwarz function \( \omega \), which is analytic in \( U \) with 
\[ \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in U) \]
such that 
\[ f(z) = g(\omega(z)) \quad (z \in U). \]
Indeed, it is known that 
\[ f(z) \prec g(z) \implies f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U). \]
Furthermore, if the function \( g \) is univalent in \( U \), then we have the following equivalence:
\[ f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(U) \subset g(U). \]

We recall the general Hurwitz-Lerch Zeta function \( \Phi(z, s, a) \) defined by (cf., e.g., [30, p. 121 et sep.])
\[ \Phi(z, s, a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s}, \]
where, as usual, 
\[ \mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N} \quad (\mathbb{Z} := \{0, \pm1, \pm2, \ldots\}; \mathbb{N} := \{1, 2, 3, \ldots\}). \]
Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function \( \Phi(z, s, a) \) can be found in the recent investigations by (for example) Choi and Srivastava [9], Ferreira and López [11], Garg et al. [12], Lin and Srivastava [13], Lin et al. [14] and Luo and Srivastava [17].

In 2007, Srivastava and Attiya [29] (see also Răducanu and Srivastava [25], Liu [16] and Prajapat and Goyal [24]) introduced and investigated the linear operator 
\[ J_{s, b}(f) : A \longrightarrow A \]
defined in terms of the Hadamard product (or convolution) by 
\[ J_{s, b}(f)(z) := G_{s, b}(z) \ast f(z) \quad (z \in U; \ b \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ s \in \mathbb{C}; \ f \in A), \quad (1.2) \]
where, for convenience, 
\[ G_{s, b}(z) := (1 + b)^s[\Phi(z, s, b) - b^{-s}] \quad (z \in U). \quad (1.3) \]
It is easy to observe from (1.2) and (1.3) that 
\[ J_{s, b}(f)(z) = z + \sum_{k=2}^{\infty} \frac{1+b}{k+b}^{s} a_k z^k. \]
Motivated essentially by the Srivastava-Attiya operator, Al-Shaqsi and Darus [4] (see also Darus and Al-Shaqsi [10]) introduced and investigated the integral operator 
\[ J_{s, b}^{\lambda, \mu}(f)(z) = z + \sum_{k=2}^{\infty} \left( \frac{1+b}{k+b} \right)^{s} \frac{\lambda!(k+\mu-2)!}{(\mu-2)!(k+\lambda-1)!} a_k z^k \quad (z \in U), \quad (1.4) \]
where (and throughout this paper unless otherwise mentioned) the parameters \( s, b, \mu \) and \( \lambda \) are constrained as follows:
\[ s \in \mathbb{C}; \ b \in \mathbb{C} \setminus \mathbb{Z}_0^-; \ \mu > 0 \quad \text{and} \quad \lambda > -1. \]
We note that \( J_{s, b}^{1, 2} \) is the Srivastava-Attiya operator, and \( J_{0, b}^{\lambda, \mu} \) is the well-known Choi-Saigo-Srivastava operator (see \[ 8, 15, 28 \]).
It is readily verified from (1.4) that
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\[ z \left( J_{s,b}^{\lambda+1, \mu} f \right)'(z) = (\lambda + 1) J_{s,b}^{\lambda, \mu} f(z) - \lambda J_{s,b}^{\lambda+1, \mu} f(z), \quad (1.5) \]

\[ z \left( J_{s+1,b}^{\lambda, \mu} f \right)'(z) = (b + 1) J_{s,b}^{\lambda, \mu} f(z) - b J_{s+1,b}^{\lambda, \mu} f(z), \quad (1.6) \]

and

\[ z \left( J_{s,b}^{\lambda, \mu} f \right)'(z) = \mu J_{s,b}^{\lambda, \mu+1} f(z) - (\mu - 1) J_{s,b}^{\lambda, \mu} f(z). \quad (1.7) \]

In the present paper, we aim at proving some subordination-preserving and superordination-preserving properties associated with the operator \( J_{s,b}^{\lambda, \mu} \). Several sandwich-type results involving this operator are also derived (some recent sandwich-type results in analytic function theory can be found in \([1, 2, 3, 5, 6, 7, 22, 26, 27, 31]\) and the references cited therein).

2. Preliminary Results

To derive our main results, we need the following definitions and lemmas.

**Definition 1.** ([21]) A function \( P(z, t) (z \in \mathbb{U}; t \geq 0) \) is said to be a subordination chain if \( P(., t) \) is analytic and univalent in \( \mathbb{U} \) for all \( t \geq 0 \), \( P(z, 0) \) is continuously differentiable on \([0, \infty)\) for all \( z \in \mathbb{U} \) and \( P(z, t_1) \prec P(z, t_2) \) for all \( 0 \leq t_1 \leq t_2 \).

**Definition 2.** ([19]) Denote by \( Q \) the set of all functions \( f \) that are analytic and injective on \( \mathbb{U} - E(f) \), where \( E(f) = \{ \varepsilon \in \partial \mathbb{U} : \lim_{z \to \varepsilon} f(z) = \infty \} \), and such that \( f'(\varepsilon) \neq 0 \) for \( \varepsilon \in \partial \mathbb{U} - E(f) \). The subclass of \( Q \) for which \( f(0) = a \ (a \in \mathbb{C}) \) is denoted by \( Q(a) \).

**Lemma 1.** ([23]) The function \( P(z, t) : \mathbb{U} \times [0, \infty) \to \mathbb{C} \) of the form

\[ P(z, t) = a_1(t)z + a_2(t)z^2 + \cdots \quad (a_1(t) \neq 0; \ t \geq 0), \]

and \( \lim_{t \to \infty} |a_1(t)| = \infty \) is a subordination chain if and only if

\[ \Re \left( \frac{z \partial P/\partial z}{\partial P/\partial t} \right) > 0 \quad (z \in \mathbb{U}; \ t \geq 0). \]

**Lemma 2.** ([18]) Suppose that the function \( H : \mathbb{C}^2 \to \mathbb{C} \) satisfies the condition

\[ \Re(H(is, t)) \leq 0 \]

for all real \( s \) and for all

\[ t \leq -\frac{n(1 + s^2)}{2} \quad (n \in \mathbb{N}). \]

If the function

\[ p(z) = 1 + p_nz^n + p_{n+1}z^{n+1} + \cdots \]

is analytic in \( \mathbb{U} \) and

\[ \Re(H(p(z),zp'(z))) > 0 \quad (z \in \mathbb{U}), \]

then

\[ \Re(p(z)) > 0 \quad (z \in \mathbb{U}). \]
Lemma 3. ([19]) Let $\kappa, \gamma \in \mathbb{C}$ with $\kappa \neq 0$ and let $h \in H(\mathbb{U})$ with $h(0) = c$. If
\[
\Re(\kappa h(z) + \gamma) > 0 \quad (z \in \mathbb{U}),
\]
then the solution of the following differential equation:
\[
q(z) + \frac{zq'(z)}{\kappa q(z) + \gamma} = h(z) \quad (z \in \mathbb{U}; \ q(0) = c)
\]
is analytic in $\mathbb{U}$ and satisfies the inequality given by
\[
\Re(\kappa q(z) + \gamma) > 0 \quad (z \in \mathbb{U}).
\]

Lemma 4. ([20]) Let $p \in Q(a)$ and
\[
q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (q \neq a; \ n \in \mathbb{N})
\]
be analytic in $\mathbb{U}$. If $q$ is not subordinate to $p$, then there exists two points
\[
z_0 = r_0 e^{i\theta} \in \mathbb{U} \quad \text{and} \quad \xi_0 \in \partial\mathbb{U}\setminus E(f)
\]
such that
\[
q(U_{r_0}) \subset p(U), \quad q(z_0) = p(\xi_0) \quad \text{and} \quad z_0 q'(z_0) = m\xi_0 p'(\xi_0) \quad (m \geq n).
\]

Lemma 5. ([21]) Let $q \in H[a, 1]$ and $\phi : \mathbb{C}^2 \to \mathbb{C}$. Also set
\[
\phi(q(z), zq'(z)) \equiv h(z) \quad (z \in \mathbb{U}).
\]
Let
\[
P(z, t) := \phi(q(z), tzq'(z))
\]
be a subordination chain and $p \in H[a, 1] \cap Q(a)$. Then
\[
h(z) \prec \phi(p(z), zp'(z))
\]
implies that
\[
q(z) \prec p(z).
\]
Furthermore, if $\phi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in Q(a)$, then $q$ is the best subordinator.

3. Main Results

We begin by proving our first subordination property given by Theorem 1 below.

Theorem 1. Let $f, g \in A$ and $\mu > 0$. Further let
\[
\Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -q \quad (z \in \mathbb{U}; \ \varphi(z) := \frac{J_{s, b}^{\lambda, \mu+1}g(z)}{z}) \quad (3.1),
\]
where
\[
q := \frac{1 + \mu^2 - |1 - \mu^2|}{4\mu} \quad (3.2).
\]
Then the subordination
\[
\frac{J_{s, b}^{\lambda, \mu+1}f(z)}{z} \prec \frac{J_{s, b}^{\lambda, \mu+1}g(z)}{z}
\]
implies that
\[
\frac{J_{s, b}^{\lambda, \mu}f(z)}{z} \prec \frac{J_{s, b}^{\lambda, \mu}g(z)}{z}.
\]
Furthermore, the function $\frac{J_{s, b}^{\lambda, \mu}g(z)}{z}$ is the best dominant.
Proof. Let the functions $F$, $G$ and $Q$ be defined by

$$
F := \mathcal{J}_{s, \beta}^{\lambda, \mu} f(z) / z, \quad G := \mathcal{J}_{s, \beta}^{\lambda, \mu} g(z) / z \quad \text{and} \quad Q := 1 + zG''(z) / G'(z). \quad (3.3)
$$

We assume here, without loss of generality, that $G$ is analytic and univalent on $\mathbb{U}$ and $G'(\zeta) \neq 0 \ (|\zeta| = 1)$.

If not, then we replace $F$ and $G$ by $F(\rho z)$ and $G(\rho z)$, respectively, with $0 < \rho < 1$. These new functions have the desired properties on $\mathbb{U}$, and we can use them in the proof of our result. Therefore, the result would follow by letting $\rho \to 1$.

We first show that $\Re(Q(z)) > 0 \ (z \in \mathbb{U})$.

By virtue of (1.7) and the definitions of $G$ and $\varphi$, we know that

$$
\varphi(z) = G(z) + \frac{1}{\mu} zG'(z). \quad (3.4)
$$

Differentiating both sides of (3.4) with respect to $z$ yields

$$
\varphi'(z) = \left(1 + \frac{1}{\mu}\right) G'(z) + \frac{1}{\mu} zG''(z). \quad (3.5)
$$

Combining (3.3) and (3.5), we easily get

$$
1 + \frac{z\varphi''(z)}{\varphi'(z)} = Q(z) + \frac{zQ'(z)}{Q(z) + \mu} := h(z) \quad (z \in \mathbb{U}). \quad (3.6)
$$

It follows from (3.1) and (3.6) that

$$
\Re(h(z) + \mu) > 0 \quad (z \in \mathbb{U}). \quad (3.7)
$$

Moreover, by Lemma 3, we conclude that the differential equation (3.6) has a solution $Q \in \mathcal{H}(\mathbb{U})$ with $h(0) = Q(0) = 1$.

Let

$$
H(u, v) := u + \frac{v}{u + \mu} + \varphi,
$$

where $\varphi$ is given by (3.2). From (3.6) and (3.7), we obtain

$$
\Re(H(Q(z), zQ'(z))) > 0 \quad (z \in \mathbb{U}).
$$

To verify the condition that

$$
\Re(H(is, t)) \leq 0 \quad \left(s \in \mathbb{R}; \ t \leq -\frac{1 + s^2}{2}\right), \quad (3.8)
$$

we proceed it as follows:

$$
\Re(H(is, t)) = \Re \left( is + \frac{t}{is + \mu} + \varphi \right) = \frac{t\mu}{|\mu + is|^2} + \varphi \leq -\frac{\Psi(\mu, s)}{2|\mu + is|^2},
$$

where

$$
\Psi(\mu, s) := (\mu - 2\varphi)s^2 - 4\varphi\mu s - 2\varphi^2 + \mu. \quad (3.9)
$$

For $\varphi$ given by (3.2), we note that the coefficient of $s^2$ in the quadratic expression $\Psi(\mu, s)$ given by (3.9) is positive or equal to zero. Furthermore, we observe that the quadratic expression $\Psi(\mu, s)$ by $s$ in (3.9) is a perfect square, which implies that (3.8) holds. Thus, by Lemma 2, we conclude that

$$
\Re(Q(z)) > 0 \quad (z \in \mathbb{U}).
$$
By the definition of $Q$, we know that $G$ is convex. To prove $F \prec G$, let the function $P$ be defined by

$$P(z, t) := G(z) + \left(\frac{1 + t}{\mu}\right) z G'(z) \quad (z \in U; \ 0 \leq t < \infty). \quad (3.10)$$

Since $G$ is convex and $\mu > 0$, then

$$\frac{\partial P(z, t)}{\partial z} \big|_{z=0} = G'(0) \left(1 + \frac{1 + t}{\mu}\right) \neq 0 \quad (z \in U; \ 0 \leq t < \infty)$$

and

$$\Re\left(\frac{z \partial P(z, t)/\partial z}{\partial P(z, t)/\partial t}\right) = \Re\left(\mu + (1 + t)Q(z)\right) > 0 \quad (z \in U).$$

Therefore, by Lemma 1, we deduce that $P$ is a subordination chain. It follows from the definition of subordination chain that

$$\varphi(z) = G(z) + \frac{1}{\mu} z G'(z) = P(z, 0),$$

and

$$P(z, 0) \prec P(z, t) \quad (0 \leq t < \infty),$$

which implies that

$$P(\zeta, t) \notin P(U, 0) = \varphi(U) \quad (\zeta \in \partial U; \ 0 \leq t < \infty). \quad (3.11)$$

If $F$ is not subordinate to $G$, by Lemma 4, we know that there exist two points $z_0 \in U$ and $\zeta_0 \in \partial U$ such that

$$F(z_0) = G(\zeta_0) \quad \text{and} \quad z_0 F(z_0) = (1 + t)\zeta_0 G'(\zeta_0) \quad (0 \leq t < \infty). \quad (3.12)$$

Hence, by virtue of (1.7) and (3.12), we have

$$P(\zeta_0, t) = G(\zeta_0) + \frac{1 + t}{\mu} \zeta_0 G'(\zeta_0) = F(z_0) + \frac{1}{\mu} z_0 F'(z_0) = \frac{J_{s, b}^{\lambda+1, \mu} f(z_0)}{z_0} \in \varphi(U).$$

This contradicts to (3.11). Thus, we deduce that $F \prec G$. Considering $F = G$, we see that the function $G$ is the best dominant. This completes the proof of Theorem 1.

By similarly applying the method of proof of Theorem 1 as well as (1.5) and (1.6), we easily get the following results.

**Corollary 1.** Let $f, g \in A$ and $\lambda > -1$. Further let

$$\Re\left(1 + \frac{z \chi''(z)}{\chi'(z)}\right) > -\varpi \quad \left(z \in U; \ \chi(z) := \frac{J_{s, b}^{\lambda, \mu} g(z)}{z}\right),$$

where

$$\varpi := \frac{1 + (\lambda + 1)^2 - |1 - (\lambda + 1)^2|}{4(\lambda + 1)} \quad (3.13)$$

Then the subordination

$$\frac{J_{s, b}^{\lambda, \mu} f(z)}{z} \prec \frac{J_{s, b}^{\lambda, \mu} g(z)}{z}$$

implies that

$$\frac{J_{s, b}^{\lambda+1, \mu} f(z)}{z} \prec \frac{J_{s, b}^{\lambda+1, \mu} g(z)}{z}.$$

Furthermore, the function $\frac{J_{s, b}^{\lambda+1, \mu} g(z)}{z}$ is the best dominant.
Corollary 2. Let \( f, g \in A \) and \( b \in \mathbb{R} \setminus \mathbb{Z}_0 \) with \( b > -1 \). Further let
\[
\Re\left( 1 + \frac{z\lambda''(z)}{\lambda'(z)} \right) > -\vartheta \quad \left( z \in \mathbb{U}; \, \lambda(z) := \frac{J_{s, b} g(z)}{z} \right),
\]
where
\[
\vartheta := \frac{1 + (b + 1)^2 - |1 - (b + 1)^2|}{4(b + 1)}.
\]
Then the subordination
\[
\frac{J_{s, b} f(z)}{z} \prec \frac{J_{s, b} g(z)}{z}
\]
implies that
\[
\frac{J_{s+1, b} f(z)}{z} \prec \frac{J_{s+1, b} g(z)}{z}.
\]
Furthermore, the function \( \frac{J_{s+1, b} g(z)}{z} \) is the best dominant.

If \( f \) is subordinate to \( F \), then \( F \) is superordinate to \( f \). We now derive the following superordination result.

Theorem 2. Let \( f, g \in A_p \) and \( \mu > 0 \). Further let
\[
\Re\left( 1 + \frac{z\varphi''(z)}{\varphi'(z)} \right) > -\varrho \quad \left( z \in \mathbb{U}; \, \varphi(z) := \frac{J_{s, b} g(z)}{z} \right),
\]
where \( \varrho \) is given by (3.2). If the function \( \frac{J_{s, b} f(z)}{z} \) is univalent in \( \mathbb{U} \) and \( \frac{J_{s, b} f(z)}{z} \in Q \), then the subordination
\[
\frac{J_{s, b} f(z)}{z} \prec \frac{J_{s+1, b} f(z)}{z}
\]
implies that
\[
\frac{J_{s, b} g(z)}{z} \prec \frac{J_{s, b} f(z)}{z}.
\]
Furthermore, the function \( \frac{J_{s, b} g(z)}{z} \) is the best subordinant.

Proof. Suppose that the functions \( F \) and \( G \) and \( Q \) are defined by (3.3). By applying the similar method as in the proof of Theorem 1, we get
\[
\Re(Q(z)) > 0 \quad (z \in \mathbb{U}).
\]
Next, to arrive at our desired result, we show that \( G \prec F \). For this, we suppose that the function \( P \) be defined by (3.10). Since \( \mu > 0 \) and \( G \) is convex, by applying a similar method as in Theorem 1 we deduce that \( P \) is subordination chain. Therefore, by Lemma 5, we conclude that \( G \prec F \). Moreover, since the differential equation
\[
\varphi(z) = G(z) + \frac{1}{\mu} z G'(z) := \phi(G(z), zG'(z))
\]
has a univalent solution \( G \), it is the best subordinant. This completes the proof of Theorem 2. \( \square \)

Applying a similar proof as in Theorem 2 and using (1.5) and (1.6), the following results are easily obtained.
Corollary 3. Let \( f, g \in \mathcal{A} \) and \( \lambda > -1 \). Further let
\[
\Re \left( 1 + \frac{z \chi''(z)}{\chi'(z)} \right) > -\varpi \quad \left( z \in \mathbb{U}; \ \chi(z) := \frac{\mathcal{T}_{s, b}^\lambda \mu g(z)}{z} \right),
\]
where \( \varpi \) is given by \eqref{3.13}. If the function \( \frac{\mathcal{T}_{s, b}^\lambda \mu f}{z} \) is univalent in \( \mathbb{U} \) and \( \frac{\mathcal{T}_{s+1, b}^\lambda \mu f}{z} \in \mathcal{Q} \), then the subordination
\[
\frac{\mathcal{T}_{s, b}^\lambda \mu g(z)}{z} \prec \frac{\mathcal{T}_{s, b}^\lambda \mu f(z)}{z}
\]
implies that
\[
\frac{\mathcal{T}_{s+1, b}^\lambda \mu g(z)}{z} \prec \frac{\mathcal{T}_{s+1, b}^\lambda \mu f(z)}{z}.
\]
Furthermore, the function \( \frac{\mathcal{T}_{s+1, b}^\lambda \mu g(z)}{z} \) is the best subordinant.

Corollary 4. Let \( f, g \in \mathcal{A} \) and \( b \in \mathbb{R} \setminus \mathbb{Z} \) with \( b > -1 \). Further let
\[
\Re \left( 1 + \frac{z \chi''(z)}{\chi'(z)} \right) > -\vartheta \quad \left( z \in \mathbb{U}; \ \chi(z) := \frac{\mathcal{T}_{s, b}^\lambda \mu g(z)}{z} \right),
\]
where \( \vartheta \) is given by \eqref{3.14}. If the function \( \frac{\mathcal{T}_{s, b}^\lambda \mu f}{z} \) is univalent in \( \mathbb{U} \) and \( \frac{\mathcal{T}_{s+1, b}^\lambda \mu f}{z} \in \mathcal{Q} \), then the subordination
\[
\frac{\mathcal{T}_{s, b}^\lambda \mu g(z)}{z} \prec \frac{\mathcal{T}_{s, b}^\lambda \mu f(z)}{z}
\]
implies that
\[
\frac{\mathcal{T}_{s+1, b}^\lambda \mu g(z)}{z} \prec \frac{\mathcal{T}_{s+1, b}^\lambda \mu f(z)}{z}.
\]
Furthermore, the function \( \frac{\mathcal{T}_{s+1, b}^\lambda \mu g(z)}{z} \) is the best subordinant.

Combining the above mentioned subordination and superordination results involving the operator \( \mathcal{T}_{s, b}^\lambda \mu \), the following “sandwich-type results” are derived.

Corollary 5. Let \( f, g_k \in \mathcal{A} \) \((k = 1, 2)\) and \( \mu > 0 \). Further let
\[
\Re \left( 1 + \frac{z \varphi''_k(z)}{\varphi'_k(z)} \right) > -\varrho \quad \left( z \in \mathbb{U}; \ \varphi_k(z) := \frac{\mathcal{T}_{s, b}^\lambda \mu^+ g_k(z)}{z} \ (k = 1, 2) \right),
\]
where \( \varrho \) is given by \eqref{3.2}. If the function \( \frac{\mathcal{T}_{s, b}^\lambda \mu^+ f}{z} \) is univalent in \( \mathbb{U} \) and \( \frac{\mathcal{T}_{s+1, b}^\lambda \mu^+ f}{z} \in \mathcal{Q} \), then the subordination chain
\[
\frac{\mathcal{T}_{s, b}^\lambda \mu^+ g_1(z)}{z} \prec \frac{\mathcal{T}_{s, b}^\lambda \mu^+ f(z)}{z} \prec \frac{\mathcal{T}_{s, b}^\lambda \mu^+ g_2(z)}{z}
\]
implies that
\[
\frac{\mathcal{T}_{s, b}^\lambda \mu g_1(z)}{z} \prec \frac{\mathcal{T}_{s, b}^\lambda \mu f(z)}{z} \prec \frac{\mathcal{T}_{s, b}^\lambda \mu g_2(z)}{z}.
\]
Furthermore, the functions \( \frac{\mathcal{T}_{s, b}^\lambda \mu g_1}{z} \) and \( \frac{\mathcal{T}_{s, b}^\lambda \mu g_2}{z} \) are, respectively, the best subordinant and the best dominant.
Corollary 6. Let \( f, g_k \in A \ (k = 1, 2) \) and \( \lambda > -1 \). Further let
\[
\Re \left( 1 + \frac{z \chi''_k(z)}{\chi_k(z)} \right) > -\varpi \quad \left( z \in \mathbb{U}; \chi_k(z) := \frac{J_{s, b}^{\lambda, \mu} g_k(z)}{z} \quad (k = 1, 2) \right),
\]
where \( \varpi \) is given by (3.13). If the function \( \frac{J_{s, b}^{\lambda, \mu} f(z)}{z} \) is univalent in \( \mathbb{U} \) and \( \frac{J_{s, b}^{\lambda+1, \mu} f(z)}{z} \in Q \), then the subordination chain
\[
\frac{J_{s, b}^{\lambda+1, \mu} g_1(z)}{z} \prec \frac{J_{s, b}^{\lambda+1, \mu} f(z)}{z} \prec \frac{J_{s, b}^{\lambda+1, \mu} g_2(z)}{z}
\]
implies that
\[
\frac{J_{s, b}^{\lambda+1, \mu} g_1(z)}{z} \prec \frac{J_{s, b}^{\lambda+1, \mu} f(z)}{z} \prec \frac{J_{s, b}^{\lambda+1, \mu} g_2(z)}{z}.
\]
Furthermore, the functions \( \frac{J_{s, b}^{\lambda+1, \mu} g_1(z)}{z} \) and \( \frac{J_{s, b}^{\lambda+1, \mu} g_2(z)}{z} \) are, respectively, the best subordinant and the best dominant.

Corollary 7. Let \( f, g_k \in A \ (k = 1, 2) \) and \( b \in \mathbb{R} \setminus \mathbb{Z}_0^- \) with \( b > -1 \). Further let
\[
\Re \left( 1 + \frac{z \chi''_k(z)}{\chi_k(z)} \right) > -\vartheta \quad \left( z \in \mathbb{U}; \chi_k(z) := \frac{J_{s, b}^{\lambda, \mu} g_k(z)}{z} \quad (k = 1, 2) \right),
\]
where \( \vartheta \) is given by (3.14). If the function \( \frac{J_{s, b}^{\lambda, \mu} f(z)}{z} \) is univalent in \( \mathbb{U} \) and \( \frac{J_{s+1, b}^{\lambda, \mu} f(z)}{z} \in Q \), then the subordination chain
\[
\frac{J_{s+1, b}^{\lambda, \mu} g_1(z)}{z} \prec \frac{J_{s+1, b}^{\lambda, \mu} f(z)}{z} \prec \frac{J_{s+1, b}^{\lambda, \mu} g_2(z)}{z}
\]
implies that
\[
\frac{J_{s+1, b}^{\lambda, \mu} g_1(z)}{z} \prec \frac{J_{s+1, b}^{\lambda, \mu} f(z)}{z} \prec \frac{J_{s+1, b}^{\lambda, \mu} g_2(z)}{z}.
\]
Furthermore, the functions \( \frac{J_{s+1, b}^{\lambda, \mu} g_1(z)}{z} \) and \( \frac{J_{s+1, b}^{\lambda, \mu} g_2(z)}{z} \) are, respectively, the best subordinant and the best dominant.

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References


