THE HYERS–ULAM STABILITY FOR TWO FUNCTIONAL EQUATIONS IN A SINGLE VARIABLE

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Abstract. We apply the Luxemburg–Jung fixed point theorem in generalized metric spaces to study the Hyers–Ulam stability for two functional equations in a single variable.

1. Introduction and preliminaries

According to [8], the study of stability problems for functional equations originated from a talk of S. Ulam before the Mathematics Club of the University of Wisconsin in 1940, when he proposed the following problem:

Let \( E \) and \( E' \) be Banach spaces. Does there exist for each \( \varepsilon > 0 \) a \( \delta > 0 \) such that, to each function \( f \) from \( E \) into \( E' \) such that \( \|f(x+y) - f(x) - f(y)\| \leq \delta \) for all \( x, y \in E \) there corresponds a linear transformation \( l(x) \) of \( E \) into \( E' \) satisfying the inequality \( \|f(x) - l(x)\| \leq \varepsilon \) for all \( x \) in \( E \)?

A year later, D.H. Hyers answered this question in the affirmative. He designed as a \( \delta \)-linear transformation between two Banach spaces \( E \) and \( E' \) any mapping \( f : E \to E' \) such that

\[
\|f(x+y) - f(x) - f(y)\| < \delta(x, y \in E)
\]

and proved the following theorem, which says that the Cauchy functional equation is "stable in the sense of Hyers–Ulam":

Theorem. (cf. [8] Theorem 1) Let \( E \) and \( E' \) be Banach spaces and let \( f(x) \) be a \( \delta \)-linear transformation of \( E \) into \( E' \). Then the limit \( l(x) = \lim_{n \to \infty} f(2^n x)/2^n \) exists for each \( x \in E \), \( l(x) \) is a linear transformation, and \( \|f(x) - l(x)\| \leq \delta \)

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for all $x \in E$. Moreover $l(x)$ is the only linear transformation satisfying this inequality.

Subsequently, the result of Hyers has been generalized by considering unbounded Cauchy differences (T. Aoki [2], for additive mappings and Th.M. Rassias [19], for linear mappings). The paper of Th.M. Rassias [19] has provided a great influence in the development of the theory of stability of functional equations, see e.g., [20, 7, 9, 16, 15].

Baker ([3]) studied the stability of a nonlinear functional equation by using the Banach fixed point theorem. Recently, Radu ([18], see also [5]) pointed out that many theorems concerning the stability of functional equations are consequences of the fixed point alternative of Margolis and Diaz [14]. In 1996, G. Isac and Th.M. Rassias [11] were the first mathematicians to introduce applications of stability theory of functional equations for the proof of new fixed point theorems. The reader is referred to the book [10] for an extensive account of both old and new developments of nonlinear methods with applications to fixed point theory.

In this note we apply a fixed point theorem of Jung ([12]) to study the Hyers–Ulam stability for two functional equations in a single variable. First, we extend a theorem of Baker [3] and Agarwal et al. [1] and then we obtain a stability result (in the sense of Ulam) for a functional equation discussed in [17].

2. Fixed points in generalized metric spaces

The notion of complete generalized metric space has been introduced by Luxemburg in [13], by allowing the value $+\infty$ for the distance mapping.

If $(X, d)$ is a generalized metric space then the relation $\sim$ on $X$ defined by $x \sim y$ if and only if $d(x, y) < +\infty$ is an equivalence relation on $X$, which determines a unique decomposition (called the canonical decomposition) of $X$ into disjoint equivalence classes, $X = \cup \{X_\alpha, \alpha \in A\}$. If $d_\alpha = d|_{X_\alpha \times X_\alpha}$, then $(X, d)$ is a complete generalized metric space if and only if $(X_\alpha, d_\alpha)$ is a complete metric space for each $\alpha \in A$.

The fixed point theorems of the alternative on generalized metric spaces can be obtained from the corresponding fixed point theorems on appropriate metric spaces. Namely, see [12, Theorem 3.1], if $(X, d)$ is a generalized metric space, $X = \cup \{X_\alpha, \alpha \in A\}$ is its canonical decomposition and $T : X \to X$ is a mapping such that

$$d(T(x), T(y)) < +\infty \text{ whenever } d(x, y) < +\infty,$$

then $T$ has a fixed point if and only if $T_\alpha = T|_{X_\alpha} : X_\alpha \to X_\alpha$ has a fixed point for some $\alpha \in A$.

**Definition 2.1.** A mapping $\varphi : [0, \infty] \to [0, \infty]$ is called a generalized strict comparison function if it is nondecreasing, $\varphi(\infty) = \infty$, $\lim_{n \to \infty} \varphi^n(t) = 0$ for all $0 < t < \infty$ and $t - \varphi(t) \to \infty$ as $t \to \infty$. Let $(X, d)$ be a generalized metric space and $\varphi$ be a generalized strict comparison function. A mapping $f : X \to X$ is called a strict $\varphi$-contraction if

$$d(f(x), f(y)) \leq \varphi(d(x, y))$$

for all $x, y \in X$. 

Theorem 2.2. Let \((X, d)\) be a complete generalized metric space and \(T : X \to X\) be a strict \(\varphi\)-contraction such that \(d(x_0, T(x_0)) < +\infty\) for some \(x_0 \in X\). Then \(T\) has a unique fixed point in the set \(X_{oo} := \{y \in X, d(x_0, y) < \infty\}\) and the sequence \((T^n(x))_{n \in \mathbb{N}}\) converges to the fixed point \(x^*\) for every \(x \in Y\). Moreover, \(d(x_0, T(x_0)) \leq \delta\) implies \(d(x^*, x_0) \leq \delta_\varphi := \sup\{t > 0, t - \varphi(t) \leq \delta\}\).

Proof. Let \(X = \bigcup\{X_\alpha, \alpha \in A\}\) be the canonical decomposition of \(X\). Since \(d(x_0, T(x_0)) < +\infty\), both \(x_0\) and \(T(x_0)\) belong to the class \(X_{oo}\). On the other hand, it is easy to show that \(\varphi(t) < t\) for all \(t \in (0, \infty)\). Thus, for every \(y \in X_{oo}\),

\[
d(x_0, T(y)) \leq d(x_0, T(x_0)) + d(T(x_0), T(y))
\]

\[
\leq d(x_0, T(x_0)) + \varphi(d(x_0, y)) \leq d(x_0, T(x_0)) + d(x_0, y) < \infty
\]

that is, \(X_{oo}\) is an invariant subset for \(T\). This means that the restriction \(T_{oo} = T|_{X_{oo}}\) is a strict \(\varphi\)-contraction on the metric space \((X_{oo}, d)\) and now the conclusion follows from a well known fixed point result in metrical fixed point theory (see e.g., [21, Theorem 7.1.1] or [4, section 2.5]).

3. Hyers–Ulam stability of the nonlinear functional equation

\[f(x) = F(x, f(\eta(x)))\]

The Hyers–Ulam stability for the nonlinear functional equation

\[f(x) = F(x, f(\eta(x)))\]

where \(\eta : S \to S\) and \(F : S \times X \to X\) are given mappings is discussed in [3] and [1] (for the generalized stability of this equation see [6] and [5]). In the next theorem we slightly improve [3, Theorem 2] and from [1, Theorem 13], by considering comparison functions.

Theorem 3.1. Let \(S\) be a nonempty set and \((X, d)\) be a complete metric space. Let \(\eta : S \to S, F : S \times X \to X\). Suppose that

\[d(F(x, u), F(x, v)) \leq \varphi(d(u, v))\]

\((x \in S, u, v \in X)\),

where \(\varphi : [0, \infty] \to [0, \infty]\) is a generalized strict comparison function and let \(f : S \to X, \delta > 0\) be such that

\[d(f(x), F(x, f(\eta(x)))) \leq \delta\]

\((x \in S)\).

Then there is a unique mapping \(f_s : S \to X\) such that

\[f_s(x) = F(x, f_s(\eta(x)))\]

\((x \in S)\)

and

\[d(f(x), f_s(x)) \leq \delta_\varphi\]

\((x \in S)\)

where \(\delta_\varphi := \sup\{t : t - \varphi(t) \leq \delta\}\).

Proof. Consider the set \(Y\) of all mappings \(a\) from \(S\) to \(X\). According to [3, Theorem 2], the formula \(\rho(a, b) = \sup\{d(a(x), b(x)), x \in S\}\) defines a (generalized) complete metric on \(Y\). Next, let us define the mapping \(T\) from \(Y\) to \(Y\) as follows: for every \(a \in Y\) and \(x \in S\),

\[T(a)(x) = F(x, a(\eta(x)))\].

Then, for all \(a, b \in Y\) and \(x \in S\),

\[d(T(a)(x), T(b)(x)) = d(F(x, a(\eta(x))), F(x, b(\eta(x))))\]
\begin{align*}
&\leq \varphi(d(a(\eta(x)), b(\eta(x)))) \leq \varphi(\rho(a, b)).
\end{align*}

Therefore,
\begin{equation*}
\rho(T(a), T(b)) \leq \varphi(\rho(a, b)) \quad (a, b \in Y)
\end{equation*}

that is, \(T\) is a strict \(\varphi\)-contraction on \(Y\).

As \(d(f(x), F(x, f(\eta(x)))) \leq \delta \ (x \in S)\) means that \(\rho(f, T(f)) \leq \delta\), from Theorem 2.2 it follows that there is a unique \(f_s\) in \(Y\) such that \(f_s = T(f_s)\) and d\((f(x), f_s(x))\) \(\leq \sup\{t : t - \varphi(t) \leq \delta\} \ (x \in S)\).

4. The Hyers–Ulam stability of the equation \(\mu \circ f \circ \eta = f\)

Let \(X\) be a nonempty set, \((Y, d)\) be a metric space and \(\eta : X \to X\), \(\mu : Y \to Y\) be two given functions. In the following we deal with the Hyers–Ulam stability problem for the functional equation \(\mu \circ f \circ \eta = f\), where \(f : X \to Y\) is an unknown mapping. The Hyers–Ulam–Rassias stability of this equation has been studied in [17] and [5].

**Theorem 4.1.** Let \(X\) be a nonempty set, \((Y, d)\) be a complete metric space and \(\eta : X \to X\), \(\mu : Y \to Y\) be two given functions. Suppose that \(f : X \to Y\) satisfies
\begin{equation*}
d((\mu \circ f \circ \eta)(x), f(x)) \leq \delta \quad (x \in X),
\end{equation*}

where \(\delta\) is a given positive real number. If \(\varphi : [0, \infty) \to [0, \infty)\) is a generalized strict comparison function and
\begin{equation*}
d(\mu(u), \mu(v)) \leq \varphi(d(u, v)) \quad (u, v \in Y),
\end{equation*}

then there exists a unique mapping \(c : X \to Y\), which satisfies both the equation
\begin{equation*}
\mu \circ c \circ \eta = c
\end{equation*}

and the estimation
\begin{equation*}
d(f(x), c(x)) \leq \delta \varphi \quad (x \in X).
\end{equation*}

Moreover,
\begin{equation*}
c(x) = \lim_{n \to \infty} (\mu^n \circ f \circ \eta^n)(x) \quad (x \in X).
\end{equation*}

**Proof.** Let \(E := \{a : X \to Y\}\) and \(\rho(a, b) = \sup\{d(a(x), b(x)), x \in S\}\). For every \(f \in E\), define \(T(f) : X \to Y\) by \(T(f) = \mu \circ f \circ \eta\).

From the definition of \(T\) it follows that if \(a, b \in E\) then, for all \(x \in X\),
\begin{align*}
d(T(a)(x), T(b)(x)) &= d(\mu \circ a \circ \eta(x), \mu \circ b \circ \eta(x)) \\
&\leq \varphi(d(a(f(x)), b(f(x)))) \leq \varphi(\rho(a, b)).
\end{align*}

Therefore,
\begin{equation*}
\rho(T(a), T(b)) \leq \varphi(\rho(a, b)) \quad (a, b \in E).
\end{equation*}

As \(d((\mu \circ f \circ \eta)(x), f(x)) \leq \delta \quad (x \in X)\) means that \(\rho(f, T(f)) \leq \delta\), we can use Theorem 2.2 to conclude the proof.
References


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